Semi-cyclic holey group divisible designs with block size three*

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Abstract: In this paper we discuss the existence problem for a semi-cyclic holey group divisible design of type (n, m^t) with block size 3, which is denoted by a 3-SCHGDD of type (n, m^t). When $n = 3$, a 3-SCHGDD of type $(3, m^t)$ is equivalent to a $(3, mt; m)$-cyclic holey difference matrix, denoted by a $(3, mt; m)$-CHDM.

It is shown that there is a $(3, mt; m)$-CHDM if and only if $(t-1)m \equiv 0 \pmod 2$ and $t \geq 3$ with the exception of $m \equiv 0 \pmod 2$ and $t = 3$. When $n \geq 4$, the case of t odd is considered. It is established that if $t \equiv 1 \pmod 2$ and $n \geq 4$, then there exists a 3-SCHGDD of type (n, m^t) if and only if $t \geq 3$ and $(t-1)n(n-1)m \equiv 0 \pmod 6$ with some possible exceptions of $n = 6$ and 8. The main results in this paper have been used to construct optimal two-dimensional optical orthogonal codes with weight 3 and different auto- and cross-correlation constraints by the authors recently.

Keywords: holey group divisible design; semi-cyclic; cyclic holey difference matrix; cyclic difference family; perfect difference family

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1 Introduction

A holey group divisible design is an important combinatorial configuration. Its study has been motivated by many applications in constructing various types of combinatorial objects, for example [6, 10, 17, 22]. Throughout this paper we always assume that $I_u = \{0, 1, \ldots, u-1\}$ and denote by Z_v the additive group of integers modulo v.

Let n, m and t be positive integers. Let K be a set of positive integers. A *holey group divisible design* (HGDD) K-HGDD is a quadruple $(X, \mathcal{G}, \mathcal{H}, \mathcal{B})$ which satisfies the following properties:

(1) X is a finite set of *nmt points*;

(2) $\mathcal{G}$ is a partition of X into n subsets (called *groups*), each of size mt;

(3) $\mathcal{H}$ is another partition of X into t subsets (called *holes*), each of size nm such that $|H \cap G| = m$ for each $H \in \mathcal{H}$ and $G \in \mathcal{G}$;

(4) $\mathcal{B}$ is a set of subsets (called *blocks*) of X, each of cardinality from K, such that no block contains two distinct points of any group or any hole, but any other pair of distinct points of X occurs in exactly one block of $\mathcal{B}$.

Such a design is denote by a K-HGDD of type (n, m^t). When $m = 1$, a K-HGDD of type $(n, 1^t)$ is often said to be a *modified group divisible design*, denoted by a K-MGDD of type t^n. If $K = \{k\}$, we write a $\{k\}$-HGDD as a k-HGDD, and a $\{k\}$-MGDD as a k-MGDD.

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Assaf [5] first introduced the notion of MGDDs and settled the existence of 3-MGDDs. The existence of 4-MGDDs was investigated in [7, 8, 24]. There are also some results on 5-MGDDs in [1, 2]. Wei [36] first introduced the concept of HGDDs and gave a complete existence theorem for 3-HGDDs. The existence of 4-HGDDs has been completely settled in [12, 23]. We only quote the following result for the later use.

Theorem 1.1 [36] *There exists a* 3*-HGDD of type* (n, m^t) *if and only if* $n, t \geq 3$, $(t-1)(n-1)m \equiv 0 \pmod 2$ *and* $t(t-1)n(n-1)m^2 \equiv 0 \pmod 3$.

HGDDs can be seen as a special case of double group divisible designs (DGDDs), which are introduced in [40] to simplify Stinson's proof [27] on a recursive construction for group divisible designs. HGDDs can also be considered as a generalization of holey mutually orthogonal Latin squares (HMOLS), since a k-HGDD of type (k, m^t) is equivalent to $k-2$ HMOLSs of type m^t (see [28, 36] for details).

A way to construct k-HGDDs of type (n, m^t) is the pure and mixed difference method. Let $S = \{0, t, \ldots, (m-1)t\}$ be a subgroup of order m in Z_{mt}, and $S_l = S + l$ be a coset of S in Z_{mt}, $0 \leq l \leq t-1$. Let $X = I_n \times Z_{mt}$, $\mathcal{G} = \{\{i\} \times Z_{mt} : i \in I_n\}$, and $\mathcal{H} = \{I_n \times S_l : 0 \leq l \leq t-1\}$. Take a family $\mathcal{B}^*$ of some k-subsets (called *base blocks*) of X. For each $i, j \in I_n$ and $B \in \mathcal{B}^*$, define a multi-set $\Delta_{ij}(B) = \{x - y \pmod{mt} : (i, x), (j, y) \in B, (i, x) \neq (j, y)\}$, and a multi-set $\Delta_{ij}(\mathcal{B}^*) = \bigcup_{B \in \mathcal{B}^*} \Delta_{ij}(B)$. If

$$\Delta_{ij}(\mathcal{B}^*) = \begin{cases} Z_{mt} \setminus S, & \text{if } (i,j) \in I_n \times I_n \text{ and } i \neq j, \\ \emptyset, & \text{otherwise,} \end{cases}$$

then a k-HGDD of type (n, m^t) with the point set X, the group set $\mathcal{G}$ and the hole set $\mathcal{H}$ can be generated from $\mathcal{B}^*$. The required blocks are obtained by developing all base blocks of $\mathcal{B}^*$ by successively adding 1 to the second component of each point of these base blocks modulo mt. Usually a k-HGDD obtained by this manner is said to be a *semi-cyclic k-HGDD* and denoted by a k-SCHGDD. k-SCHGDDs were first introduced in [30] to construct optimal two-dimensional optical orthogonal codes.

Example 1.2 *Here we construct a* 3*-SCHGDD of type* $(3, 1^t)$ *on* $I_3 \times Z_t$ *for any odd integer* $t \geq 3$*. Let* $S = \{0\}$ *be a subgroup of order* 1 *in* Z_t*, and* $S_l = \{l\}$ *be a coset of* S *in* Z_t, $0 \leq l \leq t-1$*. Take the group set* $\mathcal{G} = \{\{i\} \times Z_t : i \in I_3\}$*, and the hole set* $\mathcal{H} = \{I_3 \times S_l : 0 \leq l \leq t-1\}$*. Let the base block set* $\mathcal{B}^* = \{\{(0,0), (1,x), (2,2x)\} : 1 \leq x \leq t-1\}$*. It is readily checked that for each* $(i,j) \in I_3 \times I_3$ *and* $i \neq j$, $\Delta_{ij}(\mathcal{B}^*) = Z_t \setminus \{0\}$*, and for each* $(i,j) \in I_3 \times I_3$ *and* $i = j$, $\Delta_{ij}(\mathcal{B}^*) = \emptyset$.

SCHGDDs are closely related to *cyclic holey difference matrices* (CHDMs). A $(k, wt; w)$-CHDM is a $k \times w(t-1)$ matrix $D = (d_{ij})$ with entries from Z_{wt} such that for any two distinct rows x and y, the difference list $L_{xy} = \{d_{xj} - d_{yj} : j \in I_{w(t-1)}\}$ contains each integer of $Z_{wt} \setminus S$ exactly once, where $S = \{0, t, \ldots, (w-1)t\}$ is a subgroup of order w in Z_{wt}, while any integer of S does not appear in L_{xy}. A $(k, wt; w)$-CHDM is also called a *w-regular $(wt, k, 1)$-incomplete difference matrix* and denoted by a w-regular ICDM$(k; wt)$ in [14].

Lemma 1.3 [31] *A k-SCHGDD of type* (k, w^t) *is equivalent to a* $(k, wt; w)$*-CHDM.*

Example 1.4 *Example* 1.2 *presents a* 3*-SCHGDD of type* $(3, 1^t)$ *for any odd integer* $t \geq 3$*, which yields a* $(3, t; 1)$*-CHDM* $(A_1 | A_2 | \cdots | A_{t-1})$*, where* $A_i = (0, i, 2i)^T$, $1 \leq i \leq t-1$.

In this paper we shall focus on the existence problem of 3-SCHGDDs. It is easy to see that the number of base blocks in a 3-SCHGDD of type (n, m^t) is $(t-1)n(n-1)m/6$. Hence combining the result of Theorem 1.1, we have the following necessary condition for the existence of 3-SCHGDDs.

Lemma 1.5 *If there exists a 3-SCHGDD of type (n, m^t), then $n, t \geq 3$, $(t-1)(n-1)m \equiv 0 \pmod 2$ and $(t-1)n(n-1)m \equiv 0 \pmod 6$.*

As the main result, we are to prove the following theorems, which will be established at the end of Sections 5 and 7, respectively. We remark that these two theorems have been used to construct optimal two-dimensional optical orthogonal codes with weight 3 and different auto- and cross-correlation constraints [35].

Theorem 1.6 *There exists a 3-SCHGDD of type $(3, m^t)$ (i.e., a $(3, mt; m)$-CHDM) if and only if $(t-1)m \equiv 0 \pmod 2$ and $t \geq 3$ with the exception of $m \equiv 0 \pmod 2$ and $t = 3$.*

Theorem 1.7 *Assume that $t \equiv 1 \pmod 2$ and $n \geq 4$. There exists a 3-SCHGDD of type (n, m^t) if and only if $t \geq 3$ and $(t-1)n(n-1)m \equiv 0 \pmod 6$ except when $(n, m, t) = (6, 1, 3)$, and possibly when (1) $n = 6$, $m \equiv 1, 5 \pmod 6$ and $t \equiv 3, 15 \pmod{18}$, (2) $n = 8$, $m \equiv 2, 10 \pmod{12}$ and $t \equiv 7 \pmod{12}$.*

2 Preliminaries

Now we try to make the reader familiar with some basic concepts and terminologies in combinatorial design theory adopted in this paper.

Let K be a set of positive integers. A *group divisible design* (GDD) K-GDD is a triple $(X, \mathcal{G}, \mathcal{B})$ satisfying the following properties: (1) X is a finite set of *points*; (2) $\mathcal{G}$ is a partition of X into subsets (called *groups*); (3) $\mathcal{B}$ is a set of subsets (called *blocks*) of X, each of cardinality from K, such that every 2-subset of X is either contained in exactly one block or in exactly one group, but not in both. If $\mathcal{G}$ contains u_i groups of size g_i for $1 \leq i \leq r$, then we call $g_1^{u_1} g_2^{u_2} \cdots g_r^{u_r}$ the *group type* (or *type*) of the GDD. If $K = \{k\}$, we write a $\{k\}$-GDD as a k-GDD. A K-*GDD* of type 1^v is commonly called a *pairwise balanced design*, denoted by a $(v, K, 1)$-*PBD*. When $K = \{k\}$, a pairwise balanced design is called a *balanced incomplete block design*, denoted by a $(v, k, 1)$-BIBD.

Lemma 2.1 [3]

(1) *There exists a $(v, \{3, 4\}, 1)$-PBD for any integer $v \equiv 0, 1 \pmod 3$ and $v \geq 3$ with the exception of $v = 6$.*

(2) *There exists a $(v, \{3, 4, 5\}, 1)$-PBD for any integer $v \geq 3$ with the exception of $v = 6, 8$.*

(3) *There exists a $(v, \{4, 6, 7, 9\}, 1)$-PBD for any integer $v \equiv 0, 1 \pmod 3$, $v \geq 4$ and $v \notin \{10, 12, 15, 18, 19, 24, 27\}$.*

An *automorphism* π of a GDD $(X, \mathcal{G}, \mathcal{B})$ is a permutation on X leaving $\mathcal{G}$, $\mathcal{B}$ invariant, respectively. Let H be the cyclic group generated by α under the compositions of permutations. Then all blocks of the GDD can be partitioned into some block orbits under H. Choose any fixed block from each block orbit and then call it a *base block* of this GDD under H. The number of the blocks contained in a block orbit is called the *length* of the block orbit.

A K-GDD of type m^n is said to be *cyclic*, if it admits an automorphism consisting of a cycle of length mn. A cyclic K-GDD is denoted by a K-CGDD. For a K-CGDD $(X, \mathcal{G}, \mathcal{B})$, we can always identify X with Z_{mn} and $\mathcal{G}$ with $\{\{in+j : 0 \le i \le m-1\} : 0 \le j \le n-1\}$. If the length of each block orbit in a K-CGDD of type m^n is mn, then the K-GDD is called *strictly cyclic*.

Lemma 2.2 [33] *There exists a strictly cyclic* 3*-GDD of type* m^n *if and only if*

(1) $m(n-1) \equiv 0 \pmod 6$ *and* $n \ge 4$;

(2) $n \not\equiv 2, 3 \pmod 4$ *when* $m \equiv 2 \pmod 4$.

A K-GDD of type m^n is said to be *semi-cyclic*, if it admits an automorphism which permutes the elements of each group $G \in \mathcal{G}$ in an m cycle. Such a GDD is denoted by a K-SCGDD of type m^n. For a K-SCGDD $(X, \mathcal{G}, \mathcal{B})$, we can always identify X with $I_n \times Z_m$ and $\mathcal{G}$ with $\{\{i\} \times Z_m : i \in I_n\}$. In this case the automorphism can be taken as $(i, x) \longmapsto (i, x+1) \pmod{(-, m)}$, $i \in I_n$ and $x \in Z_m$. Assume that $\mathcal{B}^*$ is the set of base blocks of a K-SCGDD of type m^n. It is easy to verify that

$$\Delta_{ij}(\mathcal{B}^*) = \begin{cases} Z_m, & \text{if } (i,j) \in I_n \times I_n \text{ and } i \ne j, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Note that no element of Z_m occurs more than once in $\Delta_{ij}(\mathcal{B}^*)$, and the length of each block orbit is m in a K-SCGDD.

Lemma 2.3 [18] *There is a* 3*-SCGDD of type* m^n *if and only if* $n \ge 3$ *and*

(1) $n \equiv 1, 3 \pmod 6$ *when* $m \equiv 1, 5 \pmod 6$;

(2) $n \equiv 1 \pmod 2$ *when* $m \equiv 3 \pmod 6$;

(3) $n \equiv 0, 1, 4, 9 \pmod{12}$ *when* $m \equiv 2, 10 \pmod{12}$;

(4) $n \equiv 0, 1 \pmod 3$, $n \ne 3$ *when* $m \equiv 4, 8 \pmod{12}$;

(5) $n \ne 3$ *when* $m \equiv 0 \pmod{12}$;

(6) $n \equiv 0, 1 \pmod 4$ *when* $m \equiv 6 \pmod{12}$.

Lemma 2.4 [31,32] *There is a* 4*-SCGDD of type* m^n *if and only if* $n \ge 4$, $m(n-1) \equiv 0 \pmod 3$ *and* $mn(n-1) \equiv 0 \pmod{12}$, *except when* $n = 4$ *or* $(m, n) \in \{(2, 10), (4, 7), (6, 5)\}$, *and possibly when* (1) $n = 5$, $m \equiv \pm 6 \pmod{36}$ *and* $m \ge 30$, (2) $n = 7$, $m \equiv \pm 4 \pmod{24}$ *and* $m \ge 20$, (3) $n = 10$, $m \equiv \pm 2 \pmod{12}$ *and* $m \ge 10$.

Lemma 2.5 [31] *Suppose that a k-SCHGDD of type* (n, m^t) *and a k-SCGDD of type* m^n *exist. Then a k-SCGDD of type* $(mt)^n$ *exists.*

We remark that k-SCGDDs are closely related to *cyclic difference matrices* (CDMs). A (k, m)-CDM is a $k \times m$ matrix $D = (d_{ij})$ with entries from Z_m such that for any two distinct rows x and y, the difference list $\{d_{xj} - d_{yj} : j \in I_m\}$ contains each integer of Z_m exactly once.

Lemma 2.6 [37] *A k-SCGDD of type* m^k *is equivalent to a* (k, m)*-CDM.*

Let g, t be positive integers and K be a set of positive integers. A $(gt, g, K, 1)$-*cyclic difference family* (briefly $(gt, g, K, 1)$-CDF) is a family $\mathscr{F}$ of subsets (called *base blocks*) of Z_{gt} such that (1) if $B \in \mathscr{F}$, then $|B| \in K$; (2) $\Delta\mathscr{F} = \bigcup_{B\in\mathscr{F}} \Delta B$ covers each element of $Z_{gt} \setminus \{0, t, 2t, \ldots, (g-1)t\}$ exactly once, where $\Delta B = \{x - y : x, y \in B, x \neq y\}$. It is clear that all the base blocks of a $(gt, g, K, 1)$-CDF constitute the base block set of a strictly cyclic K-GDD of type g^t.

Let h be a positive integer. Assume that $\mathcal{F} = \{\{0, b_{1i}, b_{2i}, \ldots, b_{k_i-1,i}\} : i = 1, 2, \ldots, r\}$ is the family of base blocks of an $(hgt, hg, K, 1)$-CDF, where $k_i \in K$. Define $ele(\mathcal{F}) = \bigcup_{i=1}^{r}\{b_{1i}, b_{2i}, \ldots, b_{k_i-1,i}\}$. The $(hgt, hg, K, 1)$-CDF is said to be *h-perfect*, denoted by an $(hgt, hg, K, 1)$-h-PDF, if $ele(\mathcal{F}) \subseteq \{a + bgt : 0 \le a \le \lfloor gt/2 \rfloor, a \not\equiv 0 \pmod{t}, b = 0, 1, \ldots, h-1\}$. As usual, when $h = 1$, a $(gt, g, K, 1)$-1-PDF is abbreviated to a $(gt, g, K, 1)$-PDF. Furthermore, if $g = 1$, a $(t, 1, K, 1)$-PDF is simply written as a $(t, K, 1)$-PDF.

To illustrate these definitions, we give the following examples, which are useful later.

Example 2.7 *There is a* $(256, 32, \{3, 5\}, 1)$*-CDF as follows:*

$\{0,1,45\}$, $\{0,7,58\}$, $\{0,17,172\}$, $\{0,33,181\}$, $\{0,38,179\}$, $\{0,25,191\}$,
$\{0,2,49\}$, $\{0,9,130\}$, $\{0,18,132\}$, $\{0,35,178\}$, $\{0,62,165\}$, $\{0,26,164\}$,
$\{0,3,46\}$, $\{0,10,177\}$, $\{0,20,189\}$, $\{0,36,183\}$, $\{0,27,186\}$, $\{0,42,55\}$,
$\{0,4,54\}$, $\{0,11,182\}$, $\{0,22,173\}$, $\{0,37,187\}$, $\{0,28,185\}$, $\{0,61,163\}$,
$\{0,5,57\}$, $\{0,12,174\}$, $\{0,23,134\}$, $\{0,30,170\}$, $\{0,39,60\}$, $\{0,15,34,161,190\}$,
$\{0,6,59\}$, $\{0,14,133\}$, $\{0,63,131\}$, $\{0,31,180\}$, $\{0,41,158\}$.

Example 2.8 *For* $(t, K) \in \{(13, \{4\}), (33, \{3, 5\}), (59, \{3, 4, 5\}), (71, \{3, 5\})\}$*, there is a* $(t, K, 1)$*-PDF as follows:*

$(t, K) = (13, \{4\})$: $\{0,1,4,6\}$.
$(t, K) = (33, \{3, 5\})$: $\{0,1,6,14,16\}$, $\{0,3,12\}$, $\{0,4,11\}$.
$(t, K) = (59, \{3, 4, 5\})$: $\{0,1,6,23,26\}$, $\{0,7,15,19,28\}$, $\{0,11,27,29\}$, $\{0,14,24\}$.
$(t, K) = (71, \{3, 5\})$: $\{0,1,3,10,27\}$, $\{0,4,15,29,35\}$, $\{0,5,28\}$, $\{0,8,30\}$,
$\{0,12,33\}$, $\{0,13,32\}$, $\{0,16,34\}$.

Example 2.9 *For* $(g, t) \in \{(8, 8), (16, 6)\}$*, there is a* $(gt, g, \{3, 5\}, 1)$*-PDF as follows:*

$(g, t) = (8, 8)$: $\{0,1,3,7,26\}$, $\{0,5,27\}$, $\{0,9,21\}$, $\{0,10,28\}$, $\{0,11,31\}$,
$\{0,13,30\}$, $\{0,14,29\}$.
$(g, t) = (16, 6)$: $\{0,1,4,11,26\}$, $\{0,8,40\}$, $\{0,13,44\}$, $\{0,16,45\}$, $\{0,19,47\}$,
$\{0,2,35\}$, $\{0,5,39\}$, $\{0,9,46\}$, $\{0,14,41\}$, $\{0,17,38\}$,
$\{0,20,43\}$.

Example 2.10 *For* $(g, t) \in \{(8, 8), (16, 6)\}$*, there is a* $(2gt, 2g, \{3, 5\}, 1)$*-2-PDF as follows:*

$(g, t) = (8, 8)$: $\{0,1,4,21,30\}$, $\{0,2,7,25,67\}$, $\{0,11,90\}$, $\{0,13,94\}$, $\{0,15,91\}$,
$\{0,22,66\}$, $\{0,6,75\}$, $\{0,10,87\}$, $\{0,12,95\}$, $\{0,14,92\}$,
$\{0,19,93\}$, $\{0,27,73\}$, $\{0,28,71\}$, $\{0,31,70\}$.
$(g, t) = (16, 6)$: $\{0,1,14,47,135\}$, $\{0,15,41,97,142\}$, $\{0,7,131\}$, $\{0,19,128\}$, $\{0,25,137\}$,
$\{0,2,11,31,118\}$, $\{0,17,40,103,140\}$, $\{0,10,44\}$, $\{0,21,143\}$, $\{0,28,141\}$,
$\{0,3,8,35,133\}$, $\{0,4,43\}$, $\{0,16,115\}$, $\{0,22,139\}$, $\{0,38,111\}$.

Example 2.11 *There is a* $(96, 16, \{3, 5\}, 1)$*-4-PDF as follows:*

$\{0,1,4,11,26\}$, $\{0,2,29\}$, $\{0,5,33\}$, $\{0,8,59\}$, $\{0,9,49\}$, $\{0,31,83\}$,
$\{0,32,82\}$, $\{0,34,53\}$, $\{0,35,55\}$, $\{0,57,73\}$, $\{0,58,75\}$.

h-Perfect difference families were first introduced by Chang and Miao [14] as a generalization of perfect difference families to establish a recursive construction for optimal optical orthogonal codes. They used a different terminology called h-perfect g-regular cyclic packing. For more information on perfect difference families, the interested reader may refer to [4, 21, 34].

3 Construction methods

In this section we shall establish some recursive constructions for k-SCHGDDs.

Construction 3.1 *If there exist a k-SCHGDD of type $(n,(gw)^t)$ and a k-SCHGDD of type (n,g^w), then there exists a k-SCHGDD of type (n,g^{wt}).*

Proof Let $S=\{0,t,\dots,(gw-1)t\}$ be a subgroup of order gw in Z_{gwt}, and $S_l=S+l$ be a coset of S in Z_{gwt}, $0\le l\le t-1$. By assumption we can construct a k-SCHGDD of type $(n,(gw)^t)$ on $I_n\times Z_{gwt}$ with the group set $\{\{i\}\times Z_{gwt}: i\in I_n\}$ and the hole set $\{I_n\times S_l: 0\le l\le t-1\}$. Denote the set of its base blocks by $\mathcal{B}^*$.

Let $S'=\{0,wt,\dots,(g-1)wt\}$ be a subgroup of order g in S, and $S'_r=S'+rt$ be a coset of S' in S, $0\le r\le w-1$. By assumption we can construct a k-SCHGDD of type (n,g^w) on $I_n\times S$ with the group set $\{\{i\}\times S: i\in I_n\}$ and the hole set $\{I_n\times S'_r: 0\le r\le w-1\}$. Denote the set of all its base blocks by $\mathcal{A}^*$.

Let $S''_j=S'+j$ be a coset of S' in Z_{gwt}, $0\le j\le wt-1$. Now we construct the required k-SCHGDD of type (n,g^{wt}) on $I_n\times Z_{gwt}$ with the group set $\{\{i\}\times Z_{gwt}: i\in I_n\}$ and the hole set $\{I_n\times S''_j: 0\le j\le wt-1\}$. It is readily checked that $\mathcal{B}^*\cup\mathcal{A}^*$ constitutes all base blocks of the required design. □

Construction 3.2 *Suppose that there exist a K-SCGDD of type g^n and an l-SCHGDD of type (k,w^t) for each $k\in K$. Then there exists an l-SCHGDD of type $(n,(gw)^t)$.*

Proof Let $\mathcal{B}^*$ be the set of base blocks of the given K-SCGDD of type g^n, which is constructed on $I_n\times Z_g$ with the group set $\{\{i\}\times Z_g: i\in I_n\}$.

Let $S=\{0,t,\dots,(w-1)t\}$ be a subgroup of order w in Z_{wt}, and $S_r=S+r$ be a coset of S in Z_{wt}, $0\le r\le t-1$. For each $B\in\mathcal{B}^*$, construct an l-SCHGDD of type $(|B|,w^t)$ on $B\times Z_{wt}$ with the group set $\{\{j\}\times Z_{wt}: j\in B\}$ and the hole set $\{B\times S_r: 0\le r\le t-1\}$. Denote the set of its base blocks by $\mathcal{A}^*_B$.

Let $S'=\{0,t,\dots,(gw-1)t\}$ be a subgroup of order gw in Z_{gwt}, and $S'_h=S'+h$ be a coset of S' in Z_{gwt}, $0\le h\le t-1$. Now we construct the required l-SCHGDD of type $(n,(gw)^t)$ on $I_n\times Z_{gwt}$ with the group set $\{\{i\}\times Z_{gwt}: i\in I_n\}$ and the hole set $\{I_n\times S'_h: 0\le h\le t-1\}$. For each $A=\{(a_1,x_1,y_1),(a_2,x_2,y_2),\dots,(a_l,x_l,y_l)\}\in\mathcal{A}^*_B$, define

$$\overline{A}=\{(a_1,y_1+wtx_1),(a_2,y_2+wtx_2),\dots,(a_l,y_l+wtx_l)\}.$$

It is readily checked that $\bigcup_{B\in\mathcal{B}^*}\{\overline{A}: A\in\mathcal{A}^*_B\}$ forms the set of base blocks of the required design. □

Construction 3.3 *Suppose that there exist a strictly cyclic K-GDD of type w^t and an l-MGDD of type k^n for each $k\in K$. Then there exists an l-SCHGDD of type (n,w^t).*

Proof Let $S=\{0,t,\dots,(w-1)t\}$ be a subgroup of order w in Z_{wt}, and $S_r=S+r$ be a coset of S in Z_{wt}, $0\le r\le t-1$. Let $\mathcal{B}^*$ be the set of base blocks of the given strictly cyclic K-GDD of type w^t, which is constructed on Z_{wt} with the group set $\{S_r: 0\le r\le t-1\}$.

For each $B \in \mathcal{B}^*$, we construct an l-MGDD of type $|B|^n$ on $I_n \times B$ with the group set $\{\{i\} \times B : i \in I_n\}$ and the hole set $\{I_n \times \{j\} : j \in B\}$. Denote the set of its blocks by $\mathcal{A}_B$.

Now we construct the required l-SCHGDD of type (n, w^t) on $I_n \times Z_{wt}$ with the group set $\{\{i\} \times Z_{wt} : i \in I_n\}$ and the hole set $\{I_n \times S_r : 0 \leq r \leq t-1\}$. It is readily checked that $\bigcup_{B\in\mathcal{B}^*} \mathcal{A}_B$ forms the set of base blocks of the required design. □

Construction 3.4 *If there exist a k-SCHGDD of type (n, w^t) and a (k, v)-CDM, then there exists a k-SCHGDD of type $(n, (wv)^t)$.*

Proof Let $\mathcal{B}^*$ be the set of base blocks of the given k-SCHGDD of type (n, w^t). Let $D = (d_{ij})$ be the given (k, v)-CDM. For each $B = \{(a_1, x_1), (a_2, x_2), \ldots, (a_k, x_k)\} \in \mathcal{B}^*$, we construct a family, $\mathcal{A}^*_B$, consisting of the following v base blocks

$$\{(a_1, x_1 + wtd_{1j}), (a_2, x_2 + wtd_{2j}), \ldots, (a_k, x_k + wtd_{kj})\},$$

where $0 \leq j \leq v-1$. Let $S = \{0, t, \ldots, (wv-1)t\}$ be a subgroup of order wv in Z_{vwt}, and $S_l = S + l$ be a coset of S in Z_{vwt}, $0 \leq l \leq t-1$. Now we construct the required k-SCHGDD of type $(n, (wv)^t)$ on $I_n \times Z_{vwt}$ with the group set $\{\{i\} \times Z_{vwt} : i \in I_n\}$ and the hole set $\{I_n \times S_l : 0 \leq l \leq t-1\}$. It is readily checked that $\bigcup_{B\in\mathcal{B}^*} \mathcal{A}^*_B$ forms the set of base blocks of the required design. □

It is well known that a (k, v)-CDM does not exist for any $v \equiv 0 \pmod 2$ (see [16]). Thus when v is even, Construction 3.4 does not work. We shall present Constructions 3.5 and 3.6 to deal with this problem.

Construction 3.5 *Let $gcd(wt, hg) = 1$. Suppose that there exists a k-SCHGDD of type (n, w^t). If there exist a k-SCHGDD of type $(n, (hw)^t)$ and a $(k, hg; h)$-CHDM, then there exists a k-SCHGDD of type $(n, (hgw)^t)$.*

Proof Let $S = \{0, t, \ldots, (w-1)t\}$ be a subgroup of order w in Z_{wt}, and $S_l = S + l$ be a coset of S in Z_{wt}, $0 \leq l \leq t-1$. By assumption we can construct a k-SCHGDD of type (n, w^t) on $I_n \times Z_{wt}$ with the group set $\{\{i\} \times Z_{wt} : i \in I_n\}$ and the hole set $\{I_n \times S_l : 0 \leq l \leq t-1\}$. Denote the set of its base blocks by $\mathcal{B}^*$.

Let $D = (d_{ij})$ be the given $(k, hg; h)$-CHDM. Then for its two distinct rows x and y, the difference list $L_{xy} = \{d_{xj} - d_{yj} : j \in I_{h(g-1)}\}$ contains each integer of $Z_{hg} \setminus H$ exactly once, where $H = \{0, g, \ldots, (h-1)g\}$ is a subgroup of order h in Z_{hg}.

Let $X = I_n \times (Z_{wt} \times Z_{hg})$. Since $\gcd(wt, hg) = 1$, X is isomorphic to $I_n \times Z_{wthg}$. We shall construct the required k-SCHGDD of type $(n, (hgw)^t)$ on X with the group set $\{\{i\}\times(Z_{wt}\times Z_{hg}) : i \in I_n\}$ and the hole set $\{I_n\times(S_l\times Z_{hg}) : 0 \leq l \leq t-1\}$ as follows (note that due to $\gcd(w, hg) = 1$, $S_0 \times Z_{hg}$ is isomorphic to Z_{whg}):

First for each base block $B = \{(a_1, x_1), (a_2, x_2), \ldots, (a_k, x_k)\} \in \mathcal{B}^*$, we construct a family, $\mathcal{A}^*_B$, consisting of the following $h(g-1)$ base blocks

$$\{(a_1, x_1, d_{1j}), (a_2, x_2, d_{2j}), \ldots, (a_k, x_k, d_{kj})\},$$

where $0 \leq j \leq h(g-1) - 1$.

Next since $\gcd(wt, hg) = 1$, by assumption we can construct a k-SCHGDD of type $(n, (hw)^t)$ on $I_n \times (Z_{wt} \times H)$ with the group set $\{\{i\} \times (Z_{wt} \times H) : i \in I_n\}$ and the hole set $\{I_n \times (S_l \times H) : 0 \leq l \leq t-1\}$. Denote the set of its base blocks by $\mathcal{C}$. It is readily checked that $(\bigcup_{B\in\mathcal{B}^*} \mathcal{A}^*_B)\cup\mathcal{C}$ forms the set of base blocks of the required design. □

Construction 3.5 has a requirement that $\gcd(wt, hg) = 1$. To relax this condition, we introduce a special kind of SCHGDD called h-perfect SCHGDD as follows, which is an analogy of an h-perfect difference family.

Suppose that $\mathcal{B}^* = \{B_i : i = 1, 2, \ldots, r\}$ is the family of base blocks of a k-SCHGDD of type $(n, (hg)^t)$ on $I_n \times Z_{hgt}$, where $B_i = \{(a_{1i}, b_{1i}), (a_{2i}, b_{2i}), \ldots, (a_{ki}, b_{ki})\}$. Take $\delta_i = \min\{b_{1i}, b_{2i}, \ldots, b_{ki}\}$ for each possible i. Then we can use $B_i - \delta = \{(a_{1i}, b_{1i} - \delta), (a_{2i}, b_{2i} - \delta), \ldots, (a_{ki}, b_{ki} - \delta)\}$ instead of B_i. So without loss of generality we can assume that each base block of $\mathcal{B}^*$ is of the form $\{(a_{1i}, b_{1i}), (a_{2i}, b_{2i}), \ldots, (a_{l_i-1,i}, b_{l_i-1,i}), (a_{l_i,i}, 0), (a_{l_i+1,i}, b_{l_i+1,i}), \ldots, (a_{ki}, b_{ki})\}$, where l_i is some value from 1 to k and $b_{ji} \not\equiv 0 \pmod{t}$ for each $1 \le i \le r$, $1 \le j \le k$ and $j \ne l_i$. Define a set

$$ele(\mathcal{B}^*) = \bigcup_{i=1}^{r} \{b_{1i}, b_{2i}, \ldots, b_{l_i-1,i}, b_{l_i+1,i}, \ldots, b_{ki}\}.$$

The k-SCHGDD of type $(n, (hg)^t)$ is said to be *h-perfect* if

$$ele(\mathcal{B}^*) \subseteq \{x + ygt :\ 0 \le x \le \lfloor gt/2 \rfloor, x \not\equiv 0 \pmod{t}, 0 \le y \le h - 1\}.$$

When $h = 1$, a 1-perfect k-SCHGDD is simply called a *perfect* k-SCHGDD.

Construction 3.6 *Suppose that there exists a perfect k-SCHGDD of type (n, w^t). If there exist an h-perfect k-SCHGDD of type $(n, (hw)^t)$ and a $(k, hg; h)$-CHDM, then there exists an hg-perfect k-SCHGDD of type $(n, (hgw)^t)$.*

Proof Suppose that $\mathcal{A}^* = \{\{(a_{1i}, b_{1i}), (a_{2i}, b_{2i}), \ldots, (a_{l_i,i}, 0), \ldots, (a_{ki}, b_{ki})\} : 1 \le i \le r\}$ is the set of base blocks of the given perfect k-SCHGDD of type (n, w^t), where l_i is some value from 1 to k, $0 \le b_{ji} \le \lfloor wt/2 \rfloor$ and $b_{ji} \not\equiv 0 \pmod{t}$ for each $1 \le i \le r$, $1 \le j \le k$ and $j \ne l_i$.

Let $\mathcal{B}^* = \{\{(c_{1i}, x_{1i} + y_{1i}wt), (c_{2i}, x_{2i} + y_{2i}wt), \ldots, (c_{l'_i,i}, 0), \ldots, (c_{ki}, x_{ki} + y_{ki}wt)\} : 1 \le i \le s\}$ be the set of base blocks of the given h-perfect k-SCHGDD of type $(n, (hw)^t)$, where l'_i is some value from 1 to k, $0 \le x_{ji} \le \lfloor wt/2 \rfloor$, $x_{ji} \not\equiv 0 \pmod{t}$ and $0 \le y_{ji} \le h - 1$ for each $1 \le i \le s$, $1 \le j \le k$ and $j \ne l'_i$.

Let $D = (d_{ij})$ be the given $(k, hg; h)$-CHDM, where $d_{ij} \in Z_{hg}$ for $1 \le i \le k$ and $1 \le j \le h(g-1)$ such that for any two distinct rows α and β, $\{d_{\alpha j} - d_{\beta j} : 1 \le j \le h(g-1)\}$ contains each integer of $Z_{hg} \setminus \{0, g, \ldots, (h-1)g\}$.

Now we construct the desired hg-perfect k-SCHGDD of type $(n, (hgw)^t)$ on $I_n \times Z_{hgwt}$, whose base blocks consists of the following two parts:

(1) For each $A_i = \{(a_{1i}, b_{1i}), (a_{2i}, b_{2i}), \ldots, (a_{l_i,i}, 0), \ldots, (a_{ki}, b_{ki})\} \in \mathcal{A}^*$, we construct a family, $\mathcal{C}^*_{A_i}$, consisting of the following $h(g-1)$ base blocks

$$\{(a_{1i}, b_{1i} + (d_{2\gamma} - d_{1\gamma})wt), (a_{2i}, b_{2i} + (d_{3\gamma} - d_{1\gamma})wt), \ldots, (a_{l_i,i}, 0), \ldots, (a_{ki}, b_{ki} + (d_{k\gamma} - d_{1\gamma})wt)\},$$

where $1 \le \gamma \le h(g-1)$ and the second coordinates are reduced modulo $hgwt$. Let $\mathcal{C}^* = \bigcup_{i=1}^{r} \mathcal{C}^*_{A_i}$. For $1 \le \rho \ne \theta \le n$, define the notation $\Delta_{\rho\theta}(\mathcal{C}^*) = \bigcup_{C \in \mathcal{C}^*} \Delta_{\rho\theta}(C)$, where $\Delta_{\rho\theta}(C) = \{x - y \pmod{hgwt} : (\rho, x), (\theta, y) \in C\}$. Then, by noting that $-\lfloor wt/2 \rfloor \le b_{ei} - b_{fi} \le \lfloor wt/2 \rfloor$ for each $1 \le i \le r$ and each admissible $1 \le e \ne f \le k$, it is readily checked that for any $1 \le \rho \ne \theta \le n$,

$$\Delta_{\rho\theta}(\mathcal{C}^*) = \pm\{p + qwt : 0 \le p \le \lfloor wt/2 \rfloor, p \not\equiv 0 \pmod{t}, q \in Z_{hg} \setminus \{0, g, \ldots, (h-1)g\}\}.$$

(2) For each $B_i = \{(c_{1i}, x_{1i} + y_{1i}wt), (c_{2i}, x_{2i} + y_{2i}wt), \ldots, (c_{l'_i,i}, 0), \ldots, (c_{ki}, x_{ki} + y_{ki}wt)\} \in \mathcal{B}^*$, we take a base block

$$\bar{B}_i = \{(c_{1i}, x_{1i} + y_{1i}wtg), (c_{2i}, x_{2i} + y_{2i}wtg), \ldots, (c_{l'_i,i}, 0), \ldots, (c_{ki}, x_{ki} + y_{ki}wtg)\}.$$

Let $\mathcal{D}^* = \{\bar{B}_i : 1 \le i \le s\}$. Then similarly it can be checked that for any $1 \le \rho \ne \theta \le n$,

$$\Delta_{\rho\theta}(\mathcal{D}^*) = \pm\{p + qwt : 0 \le p \le \lfloor wt/2 \rfloor, p \not\equiv 0 \pmod{t}, q \in \{0, g, \dots, (h-1)g\}\}.$$

Thus $\mathcal{C}^* \cup \mathcal{D}^*$ forms the set of base blocks of the required hg-perfect k-SCHGDD of type $(n, (hgw)^t)$. □

To use Construction 3.6, we need to find some perfect k-SCHGDDs of type (n, w^t). So we establish the following construction, whose proof is similar to that of Construction 3.3 (here only a routine check for the perfect property is needed).

Construction 3.7 *Suppose that there exist a $(hwt, hw, K, 1)$-h-PDF and an l-MGDD of type k^n for each $k \in K$. Then there exists a h-perfect l-SCHGDD of type $(n, (hw)^t)$.*

4 Some cyclic or h-perfect difference families

To apply Constructions 3.3 and 3.7, we need some cyclic or h-perfect difference families. Actually cyclic difference families have their own rich design-theoretical content (see [4]) and many application backgrounds (for example [19, 20]). In this section, we always assume that $[a, b]$ denotes the set of integers n such that $a \le n \le b$, and $[a, b]_e$ denotes the set of even integers in $[a, b]$.

Chen [15] investigated the existences of a $(v, \{3, 4^*\}, 1)$-PDF with exactly one block of size 4 and a $(v, \{3, 4^*, 5^*\}, 1)$-PDF with exactly one block of size 4 and exactly one block of size 5. The main method he used to obtain these PDFs is by means of the following Lemma 4.1, which is proved by using extended Skolem sequences.

Lemma 4.1 [15] *Suppose that there is a $(v, K, 1)$-PDF. Let $u \ge v$ be a positive integer. Then there exists a $(6u + v, K \cup \{3\}, 1)$-PDF for any $u \equiv 0, 1 \pmod 4$ and $v \equiv 1 \pmod 4$, or $u \equiv 0, 3 \pmod 4$ and $v \equiv 3 \pmod 4$.*

Lemma 4.2 [15]

(1) *There exists a $(v, \{3, 4^*\}, 1)$-PDF for $v \equiv 1 \pmod 6$ and $v \ge 19$.*

(2) *There exists a $(v, \{3, 4^*, 5^*\}, 1)$-PDF for $v \equiv 3 \pmod 6$ and $v \ge 39$.*

Lemma 4.3 *There exists a $(v, \{3, 4, 5\}, 1)$-PDF for $v \equiv 5 \pmod 6$ and $v \ge 59$.*

Proof When $v \in \{59, 71\}$, the conclusion follows from Example 2.8. When $v \in \{65, 77, 89, 101, 113, 125, 137, 149\}$, a $(v, \{4, 5\}, 1)$-PDF can be found in Appendix B of [20]. When $v \equiv 5 \pmod 6$, $77 < v < 455$ and $v \notin \{89, 101, 113, 125, 137, 149, 413, 419, 437, 443\}$, we have given a direct construction for a $(v, \{3, 4, 5\}, 1)$-PDF by computer, which are shown in Appendix A.

When $v \equiv 5, 11 \pmod{24}$ and $v \ge 413$, assume that $v = 6u + 59$ with $u \equiv 0, 3 \pmod 4$ and $u \ge 59$. Start from a $(59, \{3, 4, 5\}, 1)$-PDF, which exists by the first paragraph of this proof. Then apply Lemma 4.1 to obtain the required $(6u + 59, \{3, 4, 5\}, 1)$-PDF.

When $v \equiv 17, 23 \pmod{24}$ and $v \ge 455$, assume that $v = 6u + 65$ with $u \equiv 0, 1 \pmod 4$ and $u \ge 65$. Start from a $(65, \{4, 5\}, 1)$-PDF, which exists by the first paragraph of this proof. Then apply Lemma 4.1 to obtain the required $(6u + 65, \{3, 4, 5\}, 1)$-PDF. □

Lemma 4.4 *If there exists a $(2t - 1, K, 1)$-PDF, then there exists a $(2t, 2, K, 1)$-PDF.*

Proof Suppose there is a $(2t - 1, K, 1)$-PDF $\mathcal{A}$. Then $\Delta^+\mathcal{A} = \bigcup_{A \in \mathcal{A}} \{x - y : x, y \in A, x > y\} = [1, t - 1]$. Obviously $\mathcal{A}$ is also a $(2t, 2, K, 1)$-PDF. □

Lemma 4.5 *There exists a* $(2t, 2, \{3, 4, 5\}, 1)$*-PDF for each integer* $t \geq 7$ *and* $t \notin \{8, 9, 11, 12, 14, 15, 18, 21, 24, 27\}$.

Proof By Lemma 4.4, it suffices to construct the corresponding $(2t-1, \{3, 4, 5\}, 1)$-PDFs. By Example 2.8 and Lemma 4.2(1), when $t \equiv 1 \pmod 3$ and $t \geq 7$, there is a $(2t-1, \{3, 4\}, 1)$-PDF. By Example 2.8 and Lemma 4.2(2), when $t \equiv 2 \pmod 3$ and $t \geq 17$, there is a $(2t-1, \{3, 4, 5\}, 1)$-PDF. By Lemma 4.3, when $t \equiv 0 \pmod 3$ and $t \geq 30$, a $(2t-1, \{3, 4, 5\}, 1)$-PDF exists. $\square$

Lemma 4.6 [29] *If* $u \geq 2d-1$ *and* $u \notin [2d+2, 8d-5]$*, then the set* $[1, 2u+1] \setminus \{c\}$ *can be partitioned into some pairs* $\{x_i, y_i\}$*,* $1 \leq i \leq u$*, such that* $\{y_i - x_i : 1 \leq i \leq u\} = [d, d+u-1]$ *whenever* $(u, c) \equiv (0, 1), (1, d), (2, 0), (3, d+1) \pmod{(4, 2)}$.

Lemma 4.7 *If there exists a* $(v, K, 1)$*-PDF, then there exists a* $(v+6u+3, 4, K \cup \{3\}, 1)$*-PDF for any* $u \geq v$*,* $u \notin [v+3, 4v-1]$*,* $2u - v \equiv 3 \pmod 4$*, and* $(u, (2u-v+5)/4) \equiv (0, 1), (1, (v+1)/2), (2, 0), (3, (v+3)/2) \pmod{(4, 2)}$.

Proof Suppose there is a $(v, K, 1)$-PDF $\mathcal{A}$, then $\Delta^+\mathcal{A} = \bigcup_{A \in \mathcal{A}} \{x - y : x, y \in A, x > y\} = [1, (v-1)/2]$. Now apply Lemma 4.6 with $c = (2u-v+5)/4$ and $d = (v+1)/2$ (note that $(v+1)/2$ is an integer since the existence of a $(v, K, 1)$-PDF implies v is odd). We have that the set $[1, 2u+1] \setminus \{(2u-v+5)/4\}$ can be partitioned into some pairs $\{x_i, y_i\}$, $1 \leq i \leq u$, such that $\{y_i - x_i : 1 \leq i \leq u\} = [(v+1)/2, (v-1)/2+u]$.

Let $B_i = \{0, x_i + (v-1)/2 + u, y_i + (v-1)/2 + u\}$ and $\mathcal{B} = \bigcup_{1 \leq i \leq u} B_i$. It is readily checked that $\Delta^+\mathcal{B} = \bigcup_{B \in \mathcal{B}} \{x - y : x, y \in B, x > y\} = [(v+1)/2, (v+1)/2 + 3u] \setminus \{(v+6u+3)/4\}$. So $\mathcal{A} \cup \mathcal{B}$ forms the desired $(v+6u+3, 4, K \cup \{3\}, 1)$-PDF. $\square$

Remark 4.8 *Lemma* 4.7 *does not work when* $u \in [v+3, 4v-1]$*. However, from the proof of Lemma* 4.7*, we know that if one can remove the condition* $u \notin [2d+2, 8d-5]$ *in Lemma* 4.6*, then Lemma* 4.7 *also holds for* $u \in [v+3, 4v-1]$.

Lemma 4.9 *There exists a* $(4t, 4, \{3, 5\}, 1)$*-PDF for* $t \equiv 0 \pmod 2$*,* $t \geq 4$ *and* $t \neq 8$.

Proof (1) When $t \equiv 4 \pmod 6$ and $t \geq 4$, the conclusion follows from Lemma 3.5 in [33].

(2) When $t \equiv 0 \pmod 6$, $t \geq 210$ or $t = 60$, start from a $(33, \{3, 5\}, 1)$-PDF, which exists by Example 2.8. Apply Lemma 4.7 to obtain a $(36+6u, 4, \{3, 5\}, 1)$-PDF with $u \equiv 2 \pmod 4$, $u \geq 33$ and $u \notin [36, 131]$. Write $t = 9 + 3u/2$. Then a $(4t, 4, \{3, 5\}, 1)$-PDF exists.

For $t \equiv 0 \pmod 6$, $6 \leq t \leq 54$, we have given a direct construction for a $(4t, 4, \{3, 5\}, 1)$-PDF by computer, which are shown in Appendix B. For $t \equiv 0 \pmod 6$ and $66 \leq t \leq 204$, by Remark 4.8, if we can show that for each $u \in [36, 131]$ and $u \equiv 2 \pmod 4$, the set $[1, 2u+1] \setminus \{(u-14)/2\}$ can be partitioned into some pairs $\{x_i, y_i\}$, $1 \leq i \leq u$, such that $\{y_i - x_i : 1 \leq i \leq u\} = [17, 16+u]$, then take $t = 9 + 3u/2$ and a $(4t, 4, \{3, 5\}, 1)$-PDF can be obtained. By computer, we have found all these required partitions, which are shown in Appendix D.

(3) Similarly, when $t \equiv 2 \pmod 6$, $t \geq 410$ or $t = 116$, start from a $(65, \{3, 5\}, 1)$-PDF, which exists by Lemma 4.3. Apply Lemma 4.7 to obtain a $(68+6u, 4, \{3, 5\}, 1)$-PDF with $u \equiv 2 \pmod 4$, $u \geq 65$ and $u \notin [68, 259]$. Write $t = 17 + 3u/2$. Then a $(4t, 4, \{3, 5\}, 1)$-PDF exists.

For $t \equiv 2 \pmod 6$, $14 \leq t \leq 110$, we have given a direct construction for a $(4t, 4, \{3, 5\}, 1)$-PDF by computer, which are shown in Appendix C. For $t \equiv 2 \pmod 6$ and $122 \leq t \leq 404$, by Remark 4.8, if we can show that for each $u \in [68, 259]$ and $u \equiv 2 \pmod 4$, the set $[1, 2u+1] \setminus \{(u-30)/2\}$ can be partitioned into some pairs $\{x_i, y_i\}$, $1 \leq i \leq u$, such that $\{y_i - x_i : 1 \leq i \leq u\} = [33, 32+u]$, then take $t = 17 + 3u/2$ and a $(4t, 4, \{3, 5\}, 1)$-PDF can be obtained. By computer, we have found all these required partitions, which are shown in Appendix E. $\square$

Corollary 4.10 *There exists a perfect* 3*-SCHGDD of type* $(3, 4^t)$ *for* $t \equiv 0 \pmod 2$*,* $t \geq 4$ *and* $t \neq 8$*.*

Proof By Lemma 4.9, there is a $(4t, 4, \{3, 5\}, 1)$-PDF for $t \equiv 0 \pmod 2$, $t \geq 4$ and $t \neq 8$. Then apply Construction 3.7 with $h = 1$ and $w = 4$ to obtain a perfect 3-SCHGDD of type $(3, 4^t)$, where the needed 3-MGDDs of type k^3, $k \in \{3, 5\}$, are from Theorem 1.1. □

The following two lemmas are from the use of extended Langford sequences.

Lemma 4.11 [39, Lemma 1.7] *Let* $(u, d) \equiv (0, 1), (1, 1), (0, 0), (3, 0) \pmod{(4, 2)}$ *such that* $u \geq 2d - 1$*. Then* $[d, d + 3u - 1]$ *can be partitioned into triples* $\{a_i, b_i, c_i\}, 1 \leq i \leq u$*, such that* $a_i + b_i = c_i$*.*

Lemma 4.12 [39, Lemma 1.10] *For* $1 \leq d \leq 4$*, if* $(u, k) \equiv (0, 1), (1, d), (2, 0), (3, d+1) \pmod{(4, 2)}$ *such that* $u \geq 2d - 3$ *and* $(u/2)(2d - 1 - u) + 1 \leq k \leq (u/2)(u - 2d + 5) + 1$*, then* $[d, d + 3u] \backslash \{k + d + u - 1\}$ *can be partitioned into triples* $\{a_i, b_i, c_i\}, 1 \leq i \leq u$*, such that* $a_i + b_i = c_i$*.*

Lemma 4.13 *There exists a* $(8t, 8, \{3, 5\}, 1)$*-2-PDF for* $t \equiv 0 \pmod 2$ *and* $t \geq 8$*.*

Proof (1) When $t \equiv 4 \pmod 6$ and $t \geq 10$, the conclusion follows immediately from Lemma 3.6 in [33].

(2) When $t = 12$, a $(8t, 8, \{3, 5\}, 1)$-2-PDF is listed below:

$$\begin{array}{lllll} \{0,1,20\}, & \{0,3,62\}, & \{0,5,69\}, & \{0,17,70\}, & \{0,13,22,52,68\}, \\ \{0,2,10\}, & \{0,4,71\}, & \{0,6,21\}, & \{0,23,58\}, & \{0,11,18,51,65\}. \end{array}$$

When $t = 18$, a $(8t, 8, \{3, 5\}, 1)$-2-PDF is listed below:

$$\begin{array}{llllll} \{0,1,21\}, & \{0,4,27\}, & \{0,7,88\}, & \{0,10,103\}, & \{0,14,105\}, & \{0,19,34,76,104\}, \\ \{0,2,24\}, & \{0,5,82\}, & \{0,8,100\}, & \{0,11,89\}, & \{0,16,95\}, & \{0,17,30,75,101\}, \\ \{0,3,32\}, & \{0,6,31\}, & \{0,9,107\}, & \{0,12,106\}, & \{0,33,80\}, & \{0,35,83\}. \end{array}$$

When $t \equiv 0 \pmod 6$ and $t \geq 24$, take $A_1 = \{0, 4t+4, t+1, 2t-2, 6t-4\}$ and $A_2 = \{0, 4t+3, t-1, 6t-7, 2t-6\}$ as two base blocks with block size five. Let $S = ([1, 2t-1] \cup [4t+1, 6t-1]) \setminus \{t, 5t\}$ and $T = \{t-5, t-3, t-1, t+1, 2t-10, 2t-8, 2t-6, 2t-2, 4t+1, 4t+2, 4t+3, 4t+4, 5t-6, 5t-5, 5t-4, 5t-3, 6t-9, 6t-7, 6t-6, 6t-4\}$.

By Lemma 4.12 with $(d, u, k) = (3, t/3-3, t/6+1)$, the set $[3, t-6] \setminus \{t/2\}$ can be partitioned into triples $\{a'_i, b'_i, c'_i\}$, $1 \leq i \leq t/3 - 3$, such that $a'_i + b'_i = c'_i$. Taking $a_i = 2a'_i$, $b_i = 2b'_i$ and $c_i = 2c'_i$ for $1 \leq i \leq t/3 - 3$, we have that $[6, 2t-12]_e \setminus \{t\}$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $1 \leq i \leq t/3 - 3$, such that $a_i + b_i = c_i$.

Furthermore, $(S \setminus T) \setminus ([6, 2t-12]_e \setminus \{t\})$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $t/3 - 2 \leq i \leq 4t/3 - 8$, such that $a_i + b_i \equiv c_i \pmod{8t}$ as follows:

$$\begin{array}{ll} \{13+2j, 5t-9-j, 5t+4+j\}, & j \in [0, t/2-11]; \\ \{t+3+2j, 9t/2-3-j, 11t/2+j\}, & j \in [0, t/2-10]; \end{array}$$

$$\begin{array}{lll} \{2,9,11\}, & \{t-7, 5t+2, 6t-5\}, & \{9t/2-2, 11t/2-5, 2t-7\}, \\ \{4, 2t-15, 2t-11\}, & \{5, 11t/2-6, 11t/2-1\}, & \{9t/2-1, 11t/2-4, 2t-5\}, \\ \{3, 5t-2, 5t+1\}, & \{2t-13, 4t+5, 6t-8\}, & \{9t/2, 11t/2-3, 2t-3\}, \\ \{7, 5t-8, 5t-1\}, & \{2t-9, 4t+6, 6t-3\}, & \{9t/2+1, 11t/2-2, 2t-1\}, \\ \{1, 6t-2, 6t-1\}, & \{5t-7, 5t+3, 2t-4\}. & \end{array}$$

Then $\{\{0, a_i, c_i\} : 1 \leq i \leq 4t/3 - 8\} \cup \{A_1, A_2\}$ forms a $(8t, 8, \{3, 5\}, 1)$-2-PDF.

(3) When $t = 8$, a $(8t, 8, \{3, 5\}, 1)$-2-PDF is listed below:

$\{0,1,3,7,44\}$, $\{0,5,14\}$, $\{0,10,45\}$, $\{0,11,47\}$, $\{0,12,34\}$, $\{0,13,38\}$, $\{0,15,33\}$.

When $t = 14$, a $(8t, 8, \{3,5\}, 1)$-2-PDF is listed below:

$$\begin{array}{llllll}
\{0,1,10\}, & \{0,4,12\}, & \{0,7,81\}, & \{0,21,82\}, & \{0,15,26,58,83\}, & \{0,2,18\}, \\
\{0,5,71\}, & \{0,13,78\}, & \{0,22,59\}, & \{0,24,60\}, & \{0,3,20\}, & \{0,6,79\}, \\
\{0,19,64\}, & \{0,23,63\}, & \{0,27,62\}. & & &
\end{array}$$

When $t \equiv 2 \pmod 6$ and $t \geq 20$, take $A = \{0, 4t+2, t+1, 2t-2, 6t-1\}$ as the base block with block size five. Let $S = ([1, 2t-1] \cup [4t+1, 6t-1]) \setminus \{t, 5t\}$ and $T = \{t-3, t+1, 2t-3, 2t-2, 4t+1, 4t+2, 5t-2, 5t-1, 6t-4, 6t-1\}$.

By Lemma 4.12 with $(d, u, k) = (3, (t-5)/3, (t-2)/6)$, the set $[3, t-2] \setminus \{t/2\}$ can be partitioned into triples $\{a'_i, b'_i, c'_i\}$, $1 \leq i \leq (t-5)/3$, such that $a'_i + b'_i = c'_i$. Taking $a_i = 2a'_i$, $b_i = 2b'_i$ and $c_i = 2c'_i$ for $1 \leq i \leq (t-5)/3$, we have that $[6, 2t-4]_e \setminus \{t\}$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $1 \leq i \leq (t-5)/3$, such that $a_i + b_i = c_i$.

Furthermore, $(S \setminus T) \setminus ([6, 2t-4]_e \setminus \{t\})$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $(t-2)/3 \leq i \leq (4t-14)/3$, such that $a_i + b_i \equiv c_i \pmod{8t}$ as follows:

$$\begin{array}{ll}
\{11+2j, 5t-6-j, 5t+5+j\}, & j \in [0, t/2-9]; \\
\{t+3+2j, 9t/2-3-j, 11t/2+j\}, & j \in [0, t/2-6];
\end{array}$$

$$\begin{array}{lll}
\{4,5,9\}, & \{t-5, 5t+3, 6t-2\}, & \{5t-3, 5t-4, 2t-7\}, \\
\{3, 5t+1, 5t+4\}, & \{t-1, 9t/2-1, 11t/2-2\}, & \{9t/2-2, 11t/2-3, 2t-5\}, \\
\{7, 5t-5, 5t+2\}, & \{1, 9t/2+1, 9t/2+2\}, & \{9t/2, 11t/2-1, 2t-1\}, \\
\{2, 6t-5, 6t-3\}. & &
\end{array}$$

Then $\{\{0, a_i, c_i\} : 1 \leq i \leq (4t-14)/3\} \cup \{A\}$ forms a $(8t, 8, \{3,5\}, 1)$-2-PDF. $\Box$

Corollary 4.14 *There exists a 2-perfect 3-SCHGDD of type $(3, 8^t)$ for $t \equiv 0 \pmod 2$ and $t \geq 8$.*

Proof By Lemma 4.13, there is a $(8t, 8, \{3,5\}, 1)$-2-PDF for $t \equiv 0 \pmod 2$ and $t \geq 8$. Then apply Construction 3.7 with $h = 2$ and $w = 4$ to obtain a 2-perfect 3-SCHGDD of type $(3, 8^t)$, where the needed 3-MGDDs of type k^3, $k \in \{3,5\}$, are from Theorem 1.1. $\Box$

With similar methods to those in Lemma 4.13, making use of Lemmas 4.11 and 4.12 thoroughly, we can construct a $(4t, 4, \{3,5\}, 1)$-CDF for any $t \equiv 1 \pmod 2$, $t \geq 7$ and $t \neq 11$, and a $(16t, 16, \{3,5\}, 1)$-CDF for any $t \equiv 0 \pmod 2$ and $t \geq 4$. We have moved these proofs to Appendices F and G, respectively.

Lemma 4.15 *There exists a $(4t, 4, \{3,5\}, 1)$-CDF for $t \equiv 1 \pmod 2$, $t \geq 7$ and $t \neq 11$.*

Lemma 4.16 *There exists a $(16t, 16, \{3,5\}, 1)$-CDF for $t \equiv 0 \pmod 2$ and $t \geq 4$.*

5 Cyclic holey difference matrices

In this section we shall establish the necessary and sufficient condition for the existence of a $(3, wt; w)$-CHDM, which is equivalent to a 3-SCHGDD of type $(3, w^t)$ by Lemma 1.3.

Lemma 5.1 [38] *Let $q = 2^\alpha 3^\beta p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the prime factorization of $q \geq 4$. If $(\alpha, \beta) \neq (1,0), (0,1)$, then there is a $(4, 2q; 2)$-CHDM.*

Let d and v be integers with $1 \leq d \leq 2v+1$. A *d-extended Skolem sequence* of order v is a sequence $(a_1, a_2, \ldots, a_v)$ of v integers satisfying $\bigcup_{i=1}^{v} \{a_i, a_i - i\} = \{1, 2, \ldots, 2v+1\} \setminus \{d\}$. When $d = 2v+1$, it is simply called a *Skolem sequence* of order v. For more details on Skolem sequences, the reader may refer to [26].

Lemma 5.2 [9] *A d-extended Skolem sequence of order v exists if and only if* (i) *d is odd and $v \equiv 0,1 \pmod 4$, or* (ii) *d is even and $v \equiv 2,3 \pmod 4$.*

Lemma 5.3 *There exists a $(3,2t;2)$-CHDM for any integer $t \geq 4$.*

Proof When $t \equiv 0,1 \pmod 4$ and $t \geq 4$, a $(3,2t;2)$-CHDM is constructed as follows:

$$(A_1|A_2|\cdots|A_{t-1}|B_1|B_2|\cdots|B_{t-1}),$$

where $A_i = (0,i,a_i)^{\mathrm{T}}$ and $B_i = (0,-i,a_i-i)^{\mathrm{T}}$, $1 \leq i \leq t-1$, such that $(a_1,a_2,\ldots,a_{t-1})$ is an t-extended Skolem sequence of order $t-1$, which exists by Lemma 5.2.

When $t \in \{6,11,18,27\}$, by Lemma 5.1 there is a $(4,2t;2)$-CHDM, and deleting any row of this CHDM yields a $(3,2t;2)$-CHDM. When $t \in \{14,15\}$, we list a $(3,2t;2)$-CHDM as follows:

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&2&3&4&5&6&7&8&9&10&11&12&13&15&16&17&18&19&20&21&22&23&24&25&26&27 \\ 16&1&4&2&26&3&10&13&15&20&22&24&8&23&25&7&9&21&5&17&11&27&12&19&18&6 \end{pmatrix};$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&2&3&4&5&6&7&8&9&10&11&12&13&14&16&17&18&19&20&21&22&23&24&25&26&27&28&29 \\ 2&4&6&8&10&12&14&16&18&20&23&28&27&1&29&5&7&9&11&13&3&17&19&21&24&26&25&22 \end{pmatrix}.$$

When $t \equiv 2,3 \pmod 4$, $t \geq 7$ and $t \notin \{11,14,15,18,27\}$, by Lemma 4.5, there is a $(2t,2,\{3,4,5\},1)$-PDF, which implies a strictly cyclic $\{3,4,5\}$-GDD of type 2^t. Start from this GDD. Apply Construction 3.3 with a 3-MGDD of type k^3, $k \in \{3,4,5\}$, which exists by Theorem 1.1, to get a 3-SCHGDD of type $(3,2^t)$. It is equivalent to a $(3,2t;2)$-CHDM. □

Lemma 5.4 *There exists a $(3,4t;4)$-CHDM for any integer $t \geq 4$.*

Proof When $t \in \{5,8,11\}$, we list a $(3,4t;4)$-CHDM as follows:

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&2&3&4&6&7&8&9&11&12&13&14&16&17&18&19 \\ 2&4&6&8&12&14&17&1&19&3&7&11&9&13&16&18 \end{pmatrix};$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&2&3&4&5&6&7&9&10&11&12&13&14&15&17&18&19&20&21&22&23&25&26&27&28&29&30&31 \\ 2&4&6&9&11&10&14&18&20&22&25&27&29&1&5&3&31&7&12&15&13&19&21&23&17&26&28&30 \end{pmatrix};$$

$$\left(\begin{array}{cccccccccccccccccccc} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&2&3&4&5&6&7&8&9&10&12&13&14&15&16&17&18&19&20&21 \\ 2&4&6&8&10&12&14&16&18&24&25&28&26&31&41&43&35&38&40&39 \\ \hline 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 23&24&25&26&27&28&29&30&31&32&34&35&36&37&38&39&40&41&42&43 \\ 7&17&5&3&15&13&23&9&21&27&20&1&19&34&30&37&36&32&29&42 \end{array}\right).$$

When $t \equiv 0 \pmod 2$, $t \geq 4$ and $t \neq 8$, by Corollary 4.10, there exists a perfect 3-SCHGDD of type $(3,4^t)$, which is also a $(3,4t;4)$-CHDM by Lemma 1.3. When $t \equiv 1 \pmod 2$, $t \geq 7$ and $t \neq 11$, by Lemma 4.15, there exists a $(4t,4,\{3,5\},1)$-CDF. Then apply Construction 3.3 with a 3-MGDD of type k^3, $k \in \{3,5\}$, which exists by Theorem 1.1, to obtain a $(3,4t;4)$-CHDM. □

Lemma 5.5 *There exists a $(3,2^x t;2^x)$-CHDM for any integers $x \geq 3$ and $t \geq 5$.*

Proof For any odd integer $t \geq 5$ and $x \geq 3$, start from a 3-SCHGDD of type $(3, 1^t)$, which is equivalent to a $(3, t; 1)$-CHDM and exists by Example 1.4. Then apply Construction 3.5 with $w = 1$, $h = 2$ and $g = 2^{x-1}$ to obtain a $(3, 2^x t; 2^x)$-CHDM, where the needed 3-SCHGDD of type $(3, 2^t)$ (i.e., a $(3, 2t; 2)$-CHDM) and the needed $(3, 2^x; 2)$-CHDM are both from Lemma 5.3.

For $(x, t) = (3, 6)$, we list a $(3, 8 \times 6; 8)$-CHDM as follows:

$$\left(\begin{array}{cccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 & 10 & 11 & 13 & 14 & 15 & 16 & 17 & 19 & 20 & 21 & 22 & 23 \\
2 & 4 & 7 & 9 & 8 & 14 & 16 & 19 & 21 & 20 & 26 & 28 & 31 & 33 & 32 & 38 & 40 & 43 & 47 & 46 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
25 & 26 & 27 & 28 & 29 & 31 & 32 & 33 & 34 & 35 & 37 & 38 & 39 & 40 & 41 & 43 & 44 & 45 & 46 & 47 \\
3 & 5 & 10 & 1 & 13 & 11 & 17 & 22 & 15 & 25 & 23 & 29 & 34 & 27 & 37 & 35 & 41 & 44 & 39 & 45
\end{array}\right).$$

For any even integer $t \geq 6$, $x \in \{3, 4\}$ and $(x, t) \neq (3, 6)$, by Lemmas 4.13 and 4.16, there is a $(2^x t, 2^x, \{3, 5\}, 1)$-CDF. Then apply Construction 3.3 with a 3-MGDD of type k^3, $k \in \{3, 5\}$, which exists by Theorem 1.1, to obtain a 3-SCHGDD of type $(3, (2^x)^t)$. It is also a $(3, 2^x t; 2^x)$-CHDM.

For $t = 6$ and $x = 5$, by Example 2.10 there is a $(32 \times 6, 32, \{3, 5\}, 1)$-2-PDF. Then applying Construction 3.7 with a 3-MGDD of type k^3, $k \in \{3, 5\}$, we have a 2-perfect 3-SCHGDD of type $(3, 32^6)$. It is also a $(3, 32 \times 6; 32)$-CHDM.

For $t = 6$ and $x = 6$, by Example 2.11, there is a $(16 \times 6, 16, \{3, 5\}, 1)$-4-PDF. Then applying Construction 3.7 with a 3-MGDD of type k^3, $k \in \{3, 5\}$, we have a 4-perfect 3-SCHGDD of type $(3, 16^6)$. Start from a perfect 3-SCHGDD of type $(3, 4^6)$, which exists by Corollary 4.10. Apply Construction 3.6 with $w = h = g = 4$, we obtain a $(3, 64 \times 6; 64)$-CHDM, where the needed $(3, 16; 4)$-CHDM is from Lemma 5.4.

For $t = 6$ and $x \geq 7$, by Example 2.9, there is a $(16 \times 6, 16, \{3, 5\}, 1)$-PDF. Then applying Construction 3.7 with a 3-MGDD of type k^3, $k \in \{3, 5\}$, we have a perfect 3-SCHGDD of type $(3, 16^6)$. Now using Construction 3.6 with $w = 16$, $h = 2$ and $g = 2^{x-5}$, we obtain a $(3, 2^x \times 6; 2^x)$-CHDM, where the needed $(3, 2^{x-4}; 2)$-CHDM is from Lemma 5.3, and the needed 2-perfect 3-SCHGDD of type $(3, 32^6)$ exists by the first paragraph of this proof.

For $t = 8$ and $x = 5$, by Example 2.7, there exists a $(32 \times 8, 32, \{3, 5\}, 1)$-CDF, which implies a strictly cyclic $\{3, 5\}$-GDD of type 32^8. Start from this GDD. Apply Construction 3.3 with a 3-MGDD of type k^3, $k \in \{3, 5\}$, to obtain a $(3, 32 \times 8; 32)$-CHDM.

For $t = 8$ and $x \geq 6$, by Example 2.9, there is a $(8 \times 8, 8, \{3, 5\}, 1)$-PDF. Then applying Construction 3.7 with a 3-MGDD of type k^3, $k \in \{3, 5\}$, we have a perfect 3-SCHGDD of type $(3, 8^8)$. Similarly, by Example 2.10 there is a $(128, 16, \{3, 5\}, 1)$-2-PDF, which yields a 2-perfect 3-SCHGDD of type $(3, 16^8)$. Now using Construction 3.6 with $w = 8$, $h = 2$ and $g = 2^{x-4}$, we obtain a $(3, 2^x \times 8; 2^x)$-CHDM, where the needed $(3, 2^{x-3}; 2)$-CHDM is from Lemma 5.3.

For any even integer $t \geq 10$ and $x \geq 5$, start from a perfect 3-SCHGDD of type $(3, 4^t)$, which exists by Corollary 4.10. Then apply Construction 3.6 with $w = 4$, $h = 2$ and $g = 2^{x-3}$ to obtain a $(3, 2^x t; 2^x)$-CHDM, where the needed 2-perfect 3-SCHGDD of type $(3, 8^t)$ is from Corollary 4.14 and the needed $(3, 2^{x-2}; 2)$-CHDM is from Lemma 5.3. $\square$

Lemma 5.6 *There is no $(3, 3m; m)$-CHDM for any positive integer $m \equiv 0 \pmod 2$.*

Proof Suppose that there were a $(3, 3m; m)$-CHDM $D = (d_{ij})_{3 \times 2m}$. For each $1 \leq j \leq 2m$, let $d'_{1j} = 0$, $d'_{2j} = d_{2j} - d_{1j}$ and $d'_{3j} = d_{3j} - d_{1j}$. Obviously $D' = (d'_{ij})$ is also a $(3, 3m; m)$-CHDM. Thus without loss of generality, we can always assume that

$$D = \left(\begin{array}{ccccc:ccccc}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 4 & 7 & \cdots & 3m-2 & 2 & 5 & 8 & \cdots & 3m-1 \\
d_{31} & d_{32} & d_{33} & \cdots & d_{3m} & d_{3,m+1} & d_{3,m+2} & d_{3,m+3} & \cdots & d_{3,2m}
\end{array}\right),$$

where $\{d_{31}, d_{32}, \ldots, d_{3m}\} = \{3l+2 : 0 \le l \le m-1\}$ and $\{d_{3,m+1}, d_{3,m+2}, \ldots, d_{3,2m}\} = \{3l+1 : 0 \le l \le m-1\}$. Hence,

$$\sum_{j=1}^{m} d_{2j} = \frac{m}{2}(3m-1) \equiv -\frac{m}{2} \pmod{3m}, \tag{1}$$

$$\sum_{j=1}^{m} d_{3j} = \frac{m}{2}(3m+1) \equiv \frac{m}{2} \pmod{3m}. \tag{2}$$

Counting (2) $-$ (1), we have $\sum_{j=1}^{m}(d_{3j} - d_{2j}) \equiv m \pmod{3m}$. However, $\{d_{3j} - d_{2j} \pmod{3m} : 1 \le j \le m\} = \{3l+1 : 0 \le l \le m-1\}$. Thus

$$\sum_{j=1}^{m}(d_{3j} - d_{2j}) = \frac{m}{2}(3m-1) \equiv -\frac{m}{2} \pmod{3m}.$$

A contradiction occurs. □

Lemma 5.7 *There is no $(3, mt; m)$-CHDM for any positive integer $m \equiv 1 \pmod 2$ and $t \equiv 0 \pmod 2$.*

Proof Suppose that there were a $(3, mt; m)$-CHDM for $m \equiv 1 \pmod 2$ and $t \equiv 0 \pmod 2$. Applying Construction 3.1 and Lemma 1.3 with a $(3, m; 1)$-CHDM, which exists by Example 1.4, we have a $(3, mt; 1)$-CHDM. However, Lemma 2.5 shows that a $(3, mt; 1)$-CHDM implies a 3-SCGDD of type $(mt)^3$, which does not exist by Lemma 2.3. A contradiction occurs. □

Proof of Theorem 1.6 When $m \equiv 1 \pmod 2$ and $t \equiv 0 \pmod 2$, or $m \equiv 0 \pmod 2$ and $t = 3$, there is no $(3, mt; m)$-CHDM by Lemmas 5.6 and 5.7. Then the necessity follows from Lemma 1.5. It suffices to consider the sufficiency.

For $m \equiv 1 \pmod 2$ and $t \equiv 1 \pmod 2$, start from a $(3, t; 1)$-CHDM, which exists by Example 1.4. By Lemmas 2.3 and 2.6, there exists a $(3, m)$-CDM. Then apply Construction 3.4 and Lemma 1.3 to obtain a $(3, mt; m)$-CHDM.

For $m \equiv 2 \pmod 4$ and $t \ge 4$, start from a $(3, 2t; 2)$-CHDM, which exists by Lemma 5.3. By Lemmas 2.3 and 2.6, there exists a $(3, m/2)$-CDM. Then apply Construction 3.4 and Lemma 1.3 to obtain a $(3, mt; m)$-CHDM.

For $m \equiv 0 \pmod{12}$ and $t \ge 4$, or $m \equiv 4, 8 \pmod{12}$, $t \equiv 1 \pmod 3$ and $t \ge 4$, start with a strictly cyclic 3-GDD of type m^t from Lemma 2.2. Apply Construction 3.3 with a 3-MGDD of type 3^3, which exists by Theorem 1.1. Then we have a 3-SCHGDD of type $(3, m^t)$. It yields a $(3, mt; m)$-CHDM.

For $m \equiv 4, 8 \pmod{12}$, $t \equiv 0, 2 \pmod 3$ and $t \ge 5$, write $m = 2^x u$, where $x \ge 2$ and $u \equiv 1 \pmod 2$. Start from a $(3, 2^x t; 2^x)$-CHDM, which exists by Lemmas 5.4 and 5.5. Then apply Construction 3.4 with a $(3, u)$-CDM to obtain a $(3, mt; m)$-CHDM. □

6 The cases of $n = 4, 5, 6, 8$

In this section we shall establish the existence of 3-SCHGDDs of type (n, m^t) for $n = 4, 5, 6, 8$ and t odd.

Lemma 6.1 [30]

(1) *There exists a 3-SCHGDD of type $(4, 1^t)$ for any odd integer $t \ge 3$.*

(2) *There exists a* 3*-SCHGDD of type* $(6, 1^t)$ *for any integer* $t \equiv 1, 5 \pmod 6$ *and* $t \geq 5$.

(3) *There exists a* 3*-SCHGDD of type* $(6, 1^9)$.

Lemma 6.2 *There exists a* 3*-SCHGDD of type* $(4, m^t)$ *for any positive integer* m *and any odd integer* $t \geq 3$.

Proof For $m \equiv 0 \pmod 2$, start from a 3-SCGDD of type m^4, which exists by Lemma 2.3. By Example 1.2, there exists a 3-SCHGDD of type $(3, 1^t)$ for any odd integer $t \geq 3$. Then apply Construction 3.2 to obtain a 3-SCHGDD of type $(4, m^t)$.

For $m \equiv 1 \pmod 2$, start from a 3-SCHGDD of type $(4, 1^t)$, which exists for any odd integer $t \geq 3$ by Lemma 6.1. By Lemmas 2.3 and 2.6, there exists a $(3, m)$-CDM. Then apply Construction 3.4 to obtain a 3-SCHGDD of type $(4, m^t)$. □

Lemma 6.3 *Let* $(t-1)m \equiv 0 \pmod 6$ *and* $t \geq 3$ *be an odd integer. There exists a* 3*-SCHGDD of type* $(5, m^t)$.

Proof For $m \equiv 0 \pmod 3$ and $t \equiv 1 \pmod 2$, start from a 3-SCGDD of type m^5, which exists by Lemma 2.3. By Example 1.2, there exists a 3-SCHGDD of type $(3, 1^t)$ for any odd integer $t \geq 3$. Then apply Construction 3.2 to obtain a 3-SCHGDD of type $(5, m^t)$.

For $m = 2$ and $t \equiv 1 \pmod 6$, by Example 2.8 and Lemma 4.2, there is a $(2t-1, \{3, 4\}, 1)$-PDF, which yields a $(2t, 2, \{3, 4\}, 1)$-PDF from Lemma 4.4. Thus we have a strictly cyclic $\{3, 4\}$-GDD of type 2^t. Start from this GDD and apply Construction 3.3 with a 3-MGDD of type k^5, $k \in \{3, 4\}$, which exists by Theorem 1.1, to obtain a 3-SCHGDD of type $(5, 2^t)$. For $m \equiv 2, 10 \pmod{12}$ and $t \equiv 1 \pmod 6$, start from the resulting 3-SCHGDD of type $(5, 2^t)$, and apply Construction 3.4 with a $(3, m/2)$-CDM to obtain a 3-SCHGDD of type $(5, m^t)$.

For $m \equiv 1, 4, 5, 7, 8, 11 \pmod{12}$ and $t \equiv 1 \pmod 6$, by Lemma 2.2, we have a strictly cyclic 3-GDD of type m^t. Then apply Construction 3.3 with a 3-MGDD of type 3^5 to obtain a 3-SCHGDD of type $(5, m^t)$. □

Lemma 6.4 *There is no* 3*-SCHGDD of type* $(6, 1^3)$.

Proof Suppose that there exists a 3-SCHGDD of type $(6, 1^3)$ on $I_6 \times Z_3$ with the group set $\{\{(i, 0), (i, 1), (i, 2)\} : 0 \leq i \leq 5\}$ and the hole set $\{\{(i, j) : 0 \leq i \leq 5\} : 0 \leq j \leq 2\}$. It has 10 base blocks. Denote the set of its base blocks by $\mathcal{B}^*$. Let $\mathcal{B}^* = \{\{(a_1^{(l)}, x_1^{(l)}), (a_2^{(l)}, x_2^{(l)}), (a_3^{(l)}, x_3^{(l)})\} : 0 \leq l \leq 9\}$. Given $b, c \in I_6$ and $b \neq c$, consider the number of base blocks containing the pairs of the form $\{(b, *), (c, *)\}$. The number is 2. Now write $\mathcal{A} = \{\{a_1^{(l)}, a_2^{(l)}, a_3^{(l)}\} : 0 \leq l \leq 9\}$. We have that $\mathcal{A}$ forms a $(6, 3, 2)$-BIBD on I_6. Up to isomorphism, there is only one $(6, 3, 2)$-BIBD [25]. Hence, without loss of generality the 10 base blocks can be assumed as follows.

$$\begin{array}{lll}
\{(0,0),(1,x_2^{(0)}),(2,x_3^{(0)})\}, & \{(1,0),(2,x_2^{(1)}),(5,x_3^{(1)})\}, & \{(0,0),(2,x_2^{(2)}),(4,x_3^{(2)})\}, \\
\{(0,0),(4,x_2^{(3)}),(5,x_3^{(3)})\}, & \{(1,0),(4,x_2^{(4)}),(5,x_3^{(4)})\}, & \{(0,0),(1,x_2^{(5)}),(3,x_3^{(5)})\}, \\
\{(0,0),(3,x_2^{(6)}),(5,x_3^{(6)})\}, & \{(1,0),(3,x_2^{(7)}),(4,x_3^{(7)})\}, & \{(2,0),(3,x_2^{(8)}),(4,x_3^{(8)})\}, \\
\{(2,0),(3,x_2^{(9)}),(5,x_3^{(9)})\}. & &
\end{array}$$

Note that $\{x_2^{(l)}, x_3^{(l)}\} = \{1, 2\}$ and $x_2^{(l)} \neq x_3^{(l)}$ for each $0 \leq l \leq 9$. If $x_2^{(0)} = 1$, then one can finish the first 5 base blocks above as follows.

$$\begin{array}{lll}
\{(0,0),(1,1),(2,2)\}, & \{(1,0),(2,2),(5,1)\}, & \{(0,0),(2,1),(4,2)\}, \\
\{(0,0),(4,1),(5,2)\}, & \{(1,0),(4,1),(5,2)\}. &
\end{array}$$

So the pair $\{(4, 1), (5, 2)\}$ appears in two base blocks, a contradiction. Similar argument holds for $x_2^{(0)} = 2$. □

Lemma 6.5 *There exists a 3-SCHGDD of type $(6, m^3)$ for any integer $m \equiv 3 \pmod 6$.*

Proof When $m = 3$, let $I = \{0, 1, 2, 3, 4, \infty\}$. We here construct a 3-SCHGDD of type $(6, 3^3)$ on $I \times Z_9$ with the group set $\{\{i\} \times Z_9 : i \in I\}$ and the hole set $\{I \times \{j, 3+j, 6+j\} : 0 \le j \le 2\}$. Only initial base blocks are listed below, and all other base blocks are obtained by developing these base blocks by $(+1, -)$ modulo $(5, -)$, where $\infty + 1 = \infty$.

$$\begin{array}{lll} \{(0,0),(1,1),(3,2)\}, & \{(0,0),(1,2),(3,4)\}, & \{(0,0),(1,4),(\infty,8)\}, \\ \{(0,0),(1,5),(4,1)\}, & \{(0,0),(1,7),(\infty,5)\}, & \{(0,0),(2,8),(\infty,1)\}. \end{array}$$

When $m \ge 9$, start from the resulting 3-SCHGDD of type $(6, 3^3)$. Then apply Construction 3.4 with a $(3, m/3)$-CDM, which exists by Lemmas 2.3 and 2.6, to obtain a 3-SCHGDD of type $(6, m^3)$. □

Lemma 6.6 *There exists a 3-SCHGDD of type $(6, 1^t)$ for any odd integer $t \not\equiv 3, 15 \pmod{18}$.*

Proof When $t \equiv 1, 5 \pmod 6$ and $t \ge 5$, the conclusion follows from Lemma 6.1. When $t \equiv 9 \pmod{18}$, write $t = 3^r u$, where $u \equiv 1, 5 \pmod 6$ and $r \ge 2$.

If $u = 1$, we use induction on r. When $r = 2$, the conclusion follows from Lemma 6.1. When $r \ge 3$, assume that there exists a 3-SCHGDD of type $(6, 1^{3^{r-1}})$. By Lemma 6.5 we have a 3-SCHGDD of type $(6, (3^{r-1})^3)$. Then apply Construction 3.1 with the given 3-SCHGDD of type $(6, 1^{3^{r-1}})$, we have the required 3-SCHGDD of type $(6, 1^{3^r})$.

If $u \ge 2$, start from the resulting 3-SCHGDD of type $(6, 1^{3^r})$. Apply Construction 3.4 with a $(3, u)$-CDM, which exists by Lemmas 2.3 and 2.6, to obtain a 3-SCHGDD of type $(6, u^{3^r})$. Then making use of Construction 3.1 with a 3-SCHGDD of type $(6, 1^u)$, which exists by Lemma 6.1, we have a 3-SCHGDD of type $(6, 1^{3^r u})$. □

Cyclotomic cosets play an important role in direct constructions for SCHGDDs. Let $p \equiv 1 \pmod n$ be a prime and $\omega \in Z_p$ be a primitive element. Let C_0^n denote the multiplicative subgroup $\{w^{in} : 0 \le i < (p-1)/n\}$ of the n-th powers in Z_p and C_j^n denote the coset of C_0^n in $Z_p \setminus \{0\}$, i.e. $C_j^n = w^j \cdot C_0^n$, $0 \le j \le n-1$. The following theorem is a variation of famous Weil's theorem on character sum, which can be also seen as a corollary of Theorem 2.2 in [11].

Theorem 6.7 [13] *Let $p \equiv 1 \pmod q$ be a prime satisfying the inequality*

$$p - [\sum_{i=0}^{s-2} \binom{s}{i} (s-i-1)(q-1)^{s-i}]\sqrt{p} - sq^{s-1} > 0.$$

Then for any given s-tuple $(j_1, j_2, \ldots, j_s) \in \{0, 1, \ldots, q-1\}^s$ and any given s-tuple $(c_1, c_2, \ldots, c_s)$ of pairwise distinct elements of Z_p, there exists an element $x \in Z_p$ such that $x + c_i \in C_{j_i}^q$ for each i.

Lemma 6.8 *Let $p \ge 5$ be a prime. There exists an element $x \in Z_p$ such that $x \in C_1^2$, $x+1 \in C_1^2$ and $x - 1 \in C_0^2$.*

Proof Take $q = 2$ and $s = 3$ in Theorem 6.7. We have that if p satisfies $p - 5\sqrt{p} - 12 > 0$, which yields $p > 45$, then there exists an element $x \in Z_p$ such that $x \in C_1^2$, $x + 1 \in C_1^2$ and $x - 1 \in C_0^2$. When $5 \le p \le 43$, it is readily checked that we may take x as $(p, x) = (5,2), (7,5), (11,6), (13,5), (17,5), (19,2), (23,10), (29,2), (31,11), (37,5), (41,6), (43,2)$. □

Lemma 6.9 *There exists a 3-SCHGDD of type $(6, 2^p)$ for any prime $p \geq 3$.*

Proof Since $(p, 2) = 1$, Z_{2p} is isomorphic to $Z_2 \times Z_p$ under the mapping $\tau: x(\text{mod } 2p) \to (x(\text{mod } 2), x(\text{mod } p))$. We construct the required 3-SCHGDD of type $(6, 2^p)$ on $I_6 \times Z_2 \times Z_p$. Let $S = \{(0,0),(1,0)\}$ be a subgroup of order 2 in $Z_2 \times Z_p$, and $S_l = S + (l, l) = \{(l, l), (l+1, l)\}$ (mod $(2, p)$) be a coset of S in $Z_2 \times Z_p$, $0 \leq l \leq p-1$. Take the group set $\mathcal{G} = \{\{j\} \times Z_2 \times Z_p : j \in I_6\}$, and the hole set $\mathcal{H} = \{I_6 \times S_l : 0 \leq l \leq p-1\}$.

When $p = 3$, all base blocks of a 3-SCHGDD of type $(6, 2^3)$ are listed as follows.

$$
\begin{array}{lll}
\{(0,0,0),(1,1,1),(2,0,2)\}, & \{(0,0,0),(2,1,1),(1,0,2)\}, & \{(0,0,0),(3,1,1),(4,0,2)\}, \\
\{(0,0,0),(4,1,1),(3,0,2)\}, & \{(0,0,0),(5,1,1),(1,1,2)\}, & \{(0,0,0),(5,0,2),(1,0,1)\}, \\
\{(0,0,0),(2,0,1),(3,1,2)\}, & \{(0,0,0),(3,0,1),(2,1,2)\}, & \{(0,0,0),(4,0,1),(5,1,2)\}, \\
\{(0,0,0),(5,0,1),(4,1,2)\}, & \{(1,0,0),(3,1,1),(4,1,2)\}, & \{(1,0,0),(4,1,1),(2,0,2)\}, \\
\{(1,0,0),(5,1,1),(3,1,2)\}, & \{(1,0,0),(3,0,2),(4,0,1)\}, & \{(1,0,0),(4,0,2),(2,0,1)\}, \\
\{(1,0,0),(3,0,1),(5,1,2)\}, & \{(2,0,0),(4,1,1),(5,1,2)\}, & \{(2,0,0),(5,1,1),(3,0,2)\}, \\
\{(2,0,0),(4,0,2),(5,0,1)\}, & \{(2,0,0),(5,0,2),(3,0,1)\}. &
\end{array}
$$

When $p \geq 5$ is a prime, take $x \in Z_p$ such that $x \in C_1^2$, $x+1 \in C_1^2$ and $x-1 \in C_0^2$. This can be done by Lemma 6.8. Let ω be a primitive element of Z_p. All $10(p-1)$ base blocks of the required 3-SCHGDD of type $(6, 2^p)$ are generated by multiplying each of the following 20 base blocks by ω^{2r}, $0 \leq r < (p-1)/2$.

$$
\begin{array}{ll}
\text{if } p \equiv 1 \pmod 4: & \\
\{(0,0,0),(1,0,1),(2,1,x)\}, & \{(0,0,0),(1,1,1),(2,0,x+1)\}, \\
\{(0,0,0),(1,0,x),(3,1,1)\}, & \{(0,0,0),(1,1,x+1),(3,0,1)\}, \\
\{(0,0,0),(2,0,1),(4,1,x)\}, & \{(0,0,0),(2,1,1),(4,0,x+1)\}, \\
\{(0,0,0),(3,0,x),(5,0,1)\}, & \{(0,0,0),(3,1,x+1),(5,1,1)\}, \\
\{(0,0,0),(4,0,1),(5,0,x)\}, & \{(0,0,0),(4,1,1),(5,1,x+1)\}, \\
\{(1,0,0),(2,0,1),(5,0,x+1)\}, & \{(1,0,0),(2,0,x+1),(5,1,1)\}, \\
\{(1,0,0),(3,0,1),(4,0,x)\}, & \{(1,0,0),(3,0,x),(4,1,1)\}, \\
\{(1,0,0),(4,0,1),(5,1,x+1)\}, & \{(1,1,0),(4,0,x),(5,1,1)\}, \\
\{(2,0,1),(3,0,0),(4,0,x)\}, & \{(2,1,1),(3,0,0),(4,1,x+1)\}, \\
\{(2,0,0),(3,0,x),(5,1,1)\}, & \{(2,1,0),(3,0,x+1),(5,1,1)\}.
\end{array}
$$

$$
\begin{array}{ll}
\text{if } p \equiv 3 \pmod 4: & \\
\{(0,0,0),(1,0,1),(2,1,x)\}, & \{(0,0,0),(1,1,1),(2,0,x+1)\}, \\
\{(0,0,0),(1,0,x),(3,1,1)\}, & \{(0,0,0),(1,1,x),(3,0,-1)\}, \\
\{(0,0,0),(2,0,1),(4,1,x)\}, & \{(0,0,0),(2,1,1),(4,0,x+1)\}, \\
\{(0,0,0),(3,0,1),(5,0,x)\}, & \{(0,0,0),(3,1,x),(5,1,1)\}, \\
\{(0,0,0),(4,0,-x),(5,0,1)\}, & \{(0,0,0),(4,1,1),(5,1,x)\}, \\
\{(1,0,0),(2,0,1),(5,0,x)\}, & \{(1,0,0),(2,0,x),(5,1,1)\}, \\
\{(1,0,0),(3,0,1),(4,0,x)\}, & \{(1,0,0),(3,0,x),(4,1,1)\}, \\
\{(1,0,0),(4,0,1),(5,1,x)\}, & \{(1,1,0),(4,0,x),(5,1,1)\}, \\
\{(2,0,1),(3,0,0),(4,0,x)\}, & \{(2,1,x),(3,0,0),(4,1,1)\}, \\
\{(2,0,0),(3,0,-x),(5,1,1)\}, & \{(2,1,0),(3,0,x),(5,1,x+1)\}.
\end{array}
$$

Let $\mathcal{B}^*$ be the set of all base block of the resulting 3-SCHGDD of type $(6, 2^p)$. For $(i, j) \in I_6 \times I_6$, write $\Delta_{ij}(\mathcal{B}^*) = \bigcup_{B \in \mathcal{B}^*} \Delta_{ij}(B) = \bigcup_{B \in \mathcal{B}^*} \{(y_1 - y_2, z_1 - z_2) \pmod{(2, p)} : (i, y_1, z_1), (j, y_2, z_2) \in \mathcal{B}^*\}$. It is readily checked that

$$
\Delta_{ij}(\mathcal{B}^*) = \begin{cases} (Z_2 \times Z_p) \setminus S, & \text{if } (i,j) \in I_6 \times I_6 \text{ and } i \neq j, \\ \emptyset, & \text{otherwise.} \end{cases} \qquad \square
$$

Lemma 6.10 *There exists a 3-SCHGDD of type* $(6, 2^t)$ *for any odd integer* $t \geq 3$.

Proof Let $t = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ be its prime factorization, where $p_i \geq 3$ is a prime, $1 \leq i \leq t$. Start from a 3-SCHGDD of type $(6, 2^{p_1})$, which exists by Lemma 6.9. Apply Construction 3.4 with a $(3, q)$-CDM, $q \in \{p_1, p_2, \ldots, p_t\}$, which exists by Lemmas 2.3 and 2.6, to obtain a 3-SCHGDD of type $(6, (2q)^{p_1})$. Then apply Construction 3.1 with a 3-SCHGDD of type $(6, 2^q)$, which exists by Lemma 6.9, to obtain a 3-SCHGDD of type $(6, 2^{p_1 q})$. Repeating the above process will produce the required 3-SCHGDD of type $(6, 2^t)$ for any odd integer $t \geq 3$. □

Lemma 6.11 *There exists a 3-SCHGDD of type* $(6, m^t)$ *for any positive integer* m *and any odd integer* $t \geq 3$ *except possibly when* $m \equiv 1, 5 \pmod 6$ *and* $t \equiv 3, 15 \pmod{18}$.

Proof When $m \equiv 2 \pmod 4$, start from a 3-SCHGDD of type $(6, 2^t)$, which exists for any odd integer $t \geq 3$ by Lemma 6.10. By Lemmas 2.3 and 2.6, there exists a $(3, m/2)$-CDM. Then apply Construction 3.4 to obtain a 3-SCHGDD of type $(6, m^t)$.

When $m \equiv 0 \pmod 4$, start from a 3-SCGDD of type m^6, which exists by Lemma 2.3. By Example 1.2, there exists a 3-SCHGDD of type $(3, 1^t)$ for any odd integer $t \geq 3$. Then apply Construction 3.2 to obtain a 3-SCHGDD of type $(6, m^t)$.

When $m \equiv 1 \pmod 2$, $t \equiv 1 \pmod 2$ and $t \not\equiv 3, 15 \pmod{18}$, start from a 3-SCHGDD of type $(6, 1^t)$, which exists by Lemma 6.6. Then apply Construction 3.4 with a $(3, m)$-CDM to obtain a 3-SCHGDD of type $(6, m^t)$.

When $m \equiv 3 \pmod 6$ and $t \equiv 3, 15 \pmod{18}$, start from a 3-SCHGDD of type $(6, t^3)$, which exists by Lemma 6.5. Then applying Construction 3.1 with a 3-SCHGDD of type $(6, 3^{t/3})$, which exists by the previous paragraph, we have a 3-SCHGDD of type $(6, 3^t)$. Now start from the resulting 3-SCHGDD of type $(6, 3^t)$, and apply Construction 3.4 with a $(3, m/3)$-CDM to obtain a 3-SCHGDD of type $(6, m^t)$. □

Lemma 6.12 *Let* $(t-1)m \equiv 0 \pmod 6$ *and* $t \geq 3$ *be an odd integer. There exists a 3-SCHGDD of type* $(8, m^t)$ *except possibly when* $m \equiv 2, 10 \pmod{12}$ *and* $t \equiv 7 \pmod{12}$.

Proof When $m \equiv 0 \pmod 6$ and $t \equiv 1 \pmod 2$, start from a 3-SCGDD of type m^8, which exists by Lemma 2.3. By Example 1.2, there exists a 3-SCHGDD of type $(3, 1^t)$ for any odd integer $t \geq 3$. Then apply Construction 3.2 to obtain a 3-SCHGDD of type $(8, m^t)$.

When $m \equiv 3 \pmod 6$ and $t \equiv 1 \pmod 2$, start from a 4-SCGDD of type 3^8, which exists by Lemma 2.4. By Lemma 6.2, there exists a 3-SCHGDD of type $(4, (m/3)^t)$. Then apply Construction 3.2 to obtain a 3-SCHGDD of type $(8, m^t)$.

When $m \equiv 1, 4, 5, 7, 8, 11 \pmod{12}$ and $t \equiv 1 \pmod 6$, or $m \equiv 2, 10 \pmod{12}$ and $t \equiv 1 \pmod{12}$, take a strictly cyclic 3-GDD of type m^t from Lemma 2.2. Then apply Construction 3.3 with a 3-MGDD of type 3^8, which exists by Theorem 1.1, to obtain a 3-SCHGDD of type $(8, m^t)$. □

7 Proof of Theorem 1.7

Lemma 7.1 *There exists a 3-SCHGDD of type* $(10, 2^3)$.

Proof We here give a construction of a 3-HGDD of type $(10, 2^3)$ on I_{60} with the group set $\{\{10i + j : 0 \leq i \leq 5\} : 0 \leq j \leq 9\}$ and the hole set $\{\{i + 10j, 30 + i + 10j : 0 \leq i \leq 9\} : 0 \leq j \leq 2\}$. Let $\alpha = (0\ 10\ 20\ 30\ 40\ 50)(1\ 11\ 21\ 31\ 41\ 51) \cdots (9\ 19\ 29\ 39\ 49\ 59)$ and $\beta = (0\ 2\ 4\ 6\ 8)(1\ 3\ 5\ 7\ 9)(10\ 12\ 14\ 16\ 18)(11\ 13\ 15\ 17\ 19) \cdots (50\ 52\ 54\ 56\ 58)(51\ 53\ 55\ 57\ 59)$

be two permutations on I_{60} and G be the group generated by α and β. Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of G. Obviously this design is isomorphic to a 3-SCHGDD of type $(10, 2^3)$.

$\{0,11,22\}$, $\{0,12,21\}$, $\{0,13,25\}$, $\{0,14,51\}$, $\{0,15,28\}$, $\{0,16,24\}$, $\{0,17,53\}$, $\{0,23,41\}$, $\{0,26,43\}$, $\{0,29,45\}$, $\{0,49,55\}$, $\{1,15,23\}$.

Lemma 7.2 *There exists a 3-SCHGDD of type* $(15, 2^3)$.

Proof We here give a construction of a 3-HGDD of type $(15, 2^3)$ on I_{90} with the group set $\{\{15i+j : 0 \le i \le 5\} : 0 \le j \le 14\}$ and the hole set $\{\{i+15j, 45+i+15j : 0 \le i \le 14\} : 0 \le j \le 2\}$. Let $\alpha = (0\ 15\ 30\ 45\ 60\ 75)(1\ 16\ 31\ 46\ 61\ 76)\cdots(14\ 31\ 44\ 59\ 74\ 89)$ and $\beta = (0\ 3\ 6\ 9\ 12)(1\ 4\ 7\ 10\ 13)(2\ 5\ 8\ 11\ 14)(15\ 18\ 21\ 24\ 27)(16\ 19\ 22\ 25\ 28)(17\ 20\ 23\ 26\ 29)\cdots(75\ 78\ 81\ 84\ 87)(76\ 79\ 82\ 85\ 88)(77\ 80\ 83\ 86\ 89)$ be two permutations on I_{90} and G be the group generated by α and β. Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of G. Obviously this design is isomorphic to a 3-SCHGDD of type $(15, 2^3)$.

$\{0,16,32\}$, $\{0,17,31\}$, $\{0,18,37\}$, $\{0,20,33\}$, $\{0,21,34\}$, $\{0,22,35\}$, $\{0,23,39\}$, $\{0,24,36\}$, $\{0,25,42\}$, $\{0,26,38\}$, $\{0,29,40\}$, $\{0,41,61\}$, $\{0,43,62\}$, $\{0,44,65\}$, $\{0,64,76\}$, $\{0,67,86\}$, $\{0,68,85\}$, $\{0,70,79\}$, $\{0,71,80\}$, $\{0,73,83\}$, $\{0,74,82\}$, $\{1,19,40\}$, $\{1,23,43\}$, $\{1,32,65\}$, $\{1,34,62\}$, $\{1,37,74\}$, $\{1,41,68\}$, $\{2,20,41\}$.

Lemma 7.3 *There exists a 3-SCHGDD of type* $(18, 2^3)$.

Proof We here give a construction of a 3-HGDD of type $(18, 2^3)$ on I_{108} with the group set $\{\{18i+j : 0 \le i \le 5\} : 0 \le j \le 17\}$ and the hole set $\{\{i+18j, 54+i+18j : 0 \le i \le 17\} : 0 \le j \le 2\}$. Let $\alpha = (0\ 18\ 36\ 54\ 72\ 90)(1\ 19\ 37\ 55\ 73\ 91)\cdots(17\ 35\ 53\ 71\ 89\ 107)$ and $\beta = (0\ 6\ 12)(1\ 7\ 13)(2\ 8\ 14)(3\ 9\ 15)(4\ 10\ 16)(5\ 11\ 17)(18\ 24\ 32)(19\ 25\ 33)(20\ 26\ 34)(21\ 27\ 35)(22\ 28\ 36)(23\ 29\ 37)\cdots(90\ 96\ 102)(91\ 97\ 103)(92\ 98\ 104)(93\ 99\ 105)(94\ 100\ 106)(95\ 101\ 107)$ be two permutations on I_{108} and G be the group generated by α and β. Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of G. Obviously this design is isomorphic to a 3-SCHGDD of type $(18, 2^3)$.

$\{0,32,52\}$, $\{0,49,73\}$, $\{0,50,74\}$, $\{0,51,86\}$, $\{0,76,97\}$, $\{0,77,91\}$, $\{0,79,101\}$, $\{0,80,103\}$, $\{0,82,94\}$, $\{0,83,104\}$, $\{0,85,98\}$, $\{0,88,37\}$, $\{0,19,38\}$, $\{0,20,39\}$, $\{0,21,40\}$, $\{0,22,41\}$, $\{0,23,42\}$, $\{0,24,53\}$, $\{0,25,45\}$, $\{0,26,43\}$, $\{0,27,44\}$, $\{0,28,48\}$, $\{0,30,46\}$, $\{0,31,47\}$, $\{0,33,92\}$, $\{0,35,93\}$, $\{0,75,95\}$, $\{0,81,105\}$, $\{0,87,99\}$, $\{0,89,100\}$, $\{1,22,38\}$, $\{1,25,39\}$, $\{1,26,40\}$, $\{1,27,41\}$, $\{1,28,45\}$, $\{1,29,44\}$, $\{1,31,46\}$, $\{1,47,74\}$, $\{1,49,80\}$, $\{1,51,75\}$, $\{1,53,76\}$, $\{1,77,93\}$, $\{1,81,94\}$, $\{1,82,105\}$, $\{1,83,100\}$, $\{1,86,107\}$, $\{1,87,98\}$, $\{1,88,99\}$, $\{1,89,101\}$, $\{2,26,39\}$, $\{2,27,50\}$, $\{2,28,41\}$, $\{2,29,52\}$, $\{2,32,47\}$, $\{2,40,82\}$, $\{2,45,88\}$, $\{2,46,89\}$, $\{2,53,87\}$, $\{2,76,100\}$, $\{2,77,101\}$, $\{2,81,106\}$, $\{3,29,51\}$, $\{3,40,88\}$, $\{3,41,83\}$, $\{3,47,82\}$, $\{3,52,89\}$, $\{3,76,101\}$, $\{4,53,83\}$.

Lemma 7.4 *There exists a 3-SCHGDD of type* $(27, 2^3)$.

Proof We here give a construction of a 3-HGDD of type $(27, 2^3)$ on I_{162} with the group set $\{\{27i+j : 0 \le i \le 5\} : 0 \le j \le 26\}$ and the hole set $\{\{i+27j, 81+i+27j : 0 \le i \le 26\} : 0 \le j \le 2\}$. Let $\alpha = (0\ 27\ 54\ 81\ 108\ 135)(1\ 28\ 55\ 82\ 109\ 136)\cdots(26\ 53\ 80\ 107\ 134\ 161)$ and $\beta = (0\ 3\ 6\ 9\ 12\ 15\ 18\ 21\ 24)(1\ 4\ 7\ 10\ 13\ 16\ 19\ 22\ 25)(2\ 5\ 8\ 11\ 14\ 17\ 20\ 22\ 26)(27\ 30\ 33\ 36\ 39\ 42\ 45\ 48$

51)(28 31 34 37 40 43 46 49 52)(29 32 35 38 41 44 47 50 53) $\cdots$ (135 138 141 144 147 150 153 156 159)(136 139 142 145 148 151 154 157 160)(137 140 143 146 149 152 155 158 161) be two permutations on I_{162} and G be the group generated by α and β. Only base blocks are listed below. All other blocks are obtained by developing these base blocks under the action of G. Obviously this design is isomorphic to a 3-SCHGDD of type $(27, 2^3)$.

$\{0, 28, 56\}$,	$\{0, 29, 55\}$,	$\{0, 30, 61\}$,	$\{0, 32, 57\}$,	$\{0, 33, 58\}$,	$\{0, 34, 59\}$,
$\{0, 35, 63\}$,	$\{0, 36, 60\}$,	$\{0, 37, 66\}$,	$\{0, 38, 62\}$,	$\{0, 39, 65\}$,	$\{0, 40, 64\}$,
$\{0, 41, 70\}$,	$\{0, 42, 136\}$,	$\{0, 43, 73\}$,	$\{0, 44, 67\}$,	$\{0, 45, 68\}$,	$\{0, 46, 69\}$,
$\{0, 47, 77\}$,	$\{0, 48, 140\}$,	$\{0, 49, 71\}$,	$\{0, 72, 124\}$,	$\{0, 74, 109\}$,	$\{0, 75, 128\}$,
$\{0, 76, 111\}$,	$\{0, 110, 142\}$,	$\{0, 112, 143\}$,	$\{0, 113, 146\}$,	$\{0, 115, 148\}$,	$\{0, 116, 154\}$,
$\{0, 118, 157\}$,	$\{0, 122, 158\}$,	$\{0, 125, 145\}$,	$\{0, 130, 151\}$,	$\{0, 131, 149\}$,	$\{0, 133, 152\}$,
$\{0, 134, 155\}$,	$\{1, 35, 70\}$,	$\{1, 37, 74\}$,	$\{1, 41, 56\}$,	$\{1, 43, 59\}$,	$\{1, 46, 149\}$,
$\{1, 58, 110\}$,	$\{1, 61, 122\}$,	$\{1, 64, 112\}$,	$\{1, 65, 125\}$,	$\{1, 67, 116\}$,	$\{1, 68, 118\}$,
$\{1, 71, 119\}$,	$\{1, 134, 146\}$,	$\{2, 59, 122\}$,	$\{2, 68, 113\}$.		

Lemma 7.5 *Let $n \equiv 0, 1 \pmod 3$ and $n \geq 4$. There exists a 3-SCHGDD of type $(n, 2^3)$.*

Proof When $n \in \{4, 6\}$, the conclusion follows from Lemmas 6.2 and 6.11. When $n \in \{10, 15, 18, 27\}$, the conclusion follows from Lemmas 7.1-7.4. When $n \in \{9, 12, 24\}$, start from a 3-SCGDD of type 2^n, which exists by Lemma 2.3. By Example 1.2, there exists a 3-SCHGDD of type $(3, 1^3)$. Then apply Construction 3.2 to obtain a 3-SCHGDD of type $(n, 2^3)$. When $n \in \{7, 19\}$, start from a 4-SCGDD of type 2^n, which exists by Lemma 2.4. By Lemma 6.2, there exists a 3-SCHGDD of type $(4, 1^3)$. Then apply Construction 3.2 to obtain a 3-SCHGDD of type $(n, 2^3)$.

When $n \equiv 0, 1 \pmod 3$, $n \geq 4$ and $n \notin \{10, 12, 15, 18, 19, 24, 27\}$, start from a $\{4, 6, 7, 9\}$-SCGDD of type 1^n, which is also a $(n, \{4, 6, 7, 9\}, 1)$-PBD and exists by Lemma 2.1. Then apply Construction 3.2 with a 3-SCHGDD of type $(k, 2^3)$, $k \in \{4, 6, 7, 9\}$, to obtain a 3-SCHGDD of type $(n, 2^3)$. $\square$

Lemma 7.6 *Let $n \equiv 0, 1 \pmod 3$ and $n \geq 7$. There exists a 3-SCHGDD of type (n, m^t) for any odd integer $t \geq 3$ and any positive integer m.*

Proof For any odd integer $t \geq 5$ and any positive integer m, or $t = 3$ and $m \equiv 1 \pmod 2$, start from a $\{3, 4\}$-SCGDD of type 1^n, which is also a $(n, \{3, 4\}, 1)$-PBD and exists by Lemma 2.1. Take a 3-SCHGDD of type (k, m^t) for $k \in \{3, 4\}$, which exists from Theorem 1.6 and Lemma 6.2, and then apply Construction 3.2 to obtain a 3-SCHGDD of type (n, m^t).

For $t = 3$ and $m \equiv 0 \pmod 4$, start from a 3-SCGDD of type m^n, which exists by Lemma 2.3. By Example 1.2, there exists a 3-SCHGDD of type $(3, 1^3)$. Then apply Construction 3.2 to obtain a 3-SCHGDD of type (n, m^3).

For $t = 3$ and $m \equiv 2 \pmod 4$, start from a 3-SCHGDD of type $(n, 2^3)$, which exists by Lemma 7.5. By Lemmas 2.3 and 2.6, there exists a $(3, m/2)$-CDM. Then apply Construction 3.4 to obtain a 3-SCHGDD of type (n, m^3). $\square$

Lemma 7.7 *Let $n \equiv 2 \pmod 3$ and $n \geq 11$. Let $(t-1)m \equiv 0 \pmod 6$ and $t \geq 3$ be an odd integer. There exists a 3-SCHGDD of type (n, m^t).*

Proof When $(t-1)m \equiv 0 \pmod 6$, $t \geq 4$ is an odd integer and m is a positive integer, or $t = 3$ and $m \equiv 3 \pmod 6$, start from a $\{3, 4, 5\}$-SCGDD of type 1^n, which is also a $(n, \{3, 4, 5\}, 1)$-PBD

and exists by Lemma 2.1. Take a 3-SCHGDD of type (k, m^t) for $k \in \{3, 4, 5\}$ from Theorem 1.6, Lemmas 6.2 and 6.3. Then apply Construction 3.2 to obtain a 3-SCHGDD of type (n, m^t).

When $t = 3$ and $m \equiv 0 \pmod{12}$, start from a 3-SCGDD of type m^n, which exists by Lemma 2.3. By Example 1.2, there exists a 3-SCHGDD of type $(3, 1^3)$. Then apply Construction 3.2 to obtain a 3-SCHGDD of type (n, m^3).

When $t = 3$ and $m = 6$, start from a 4-SCGDD of type 6^n, which exists by Lemma 2.4. By Lemma 6.1, there exists a 3-SCHGDD of type $(4, 1^3)$. Then apply Construction 3.2 to obtain a 3-SCHGDD of type $(n, 6^3)$. When $t = 3$ and $m \equiv 6 \pmod{12}$, start from the resulting 3-SCHGDD of type $(n, 6^3)$. By Lemmas 2.3 and 2.6, there exists a $(3, m/6)$-CDM. Then apply Construction 3.4 to obtain a 3-SCHGDD of type (n, m^3). □

Proof of Theorem 1.7 Combining the results of Lemmas 1.5, 6.2, 6.3, 6.4, 6.11, 6.12, 7.6 and 7.7, we complete the proof. □

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Semi-cyclic holey group divisible designs with block size three: Appendix

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This appendix is a supplement to Lemmas 4.3, 4.9, 4.15 and 4.16 of the paper *Semi-cyclic holey group divisible designs with block size three*, where Section A is for Lemma 4.3, Sections B-E are for Lemma 4.9, Section F is for Lemma 4.15 and Section G is for Lemma 4.16.

A Small $(v, \{3,4,5\}, 1)$-PDFs for $v \equiv 5 \pmod 6$

Here we list all the base blocks of a $(v, \{3,4,5\}, 1)$-PDF for $77 < v < 455$ and $v \notin \{89, 101, 113, 125, 137, 149, 413, 419, 437, 443\}$.

- $v = 83$:
$\{0,1,3,18,38\}$ $\{0,4,25,33\}$ $\{0,7,30,39\}$ $\{0,13,24,40\}$ $\{0,6,34\}$ $\{0,5,19,31,41\}$
- $v = 95$:
$\{0,2,9,21,46\}$ $\{0,1,31,36\}$ $\{0,6,17,45\}$ $\{0,8,32,42\}$ $\{0,18,40\}$ $\{0,4,20,33,47\}$ $\{0,3,26,41\}$
- $v = 107$:
$\{0,1,11,34,51\}$ $\{0,4,19,49\}$ $\{0,7,28,53\}$ $\{0,16,29,47\}$ $\{0,5,37\}$ $\{0,2,14,38,41\}$ $\{0,6,26,48\}$
$\{0,8,43,52\}$
- $v = 119$:
$\{0,5,14,31,58\}$ $\{0,1,21,56\}$ $\{0,3,42,46\}$ $\{0,8,24,57\}$ $\{0,15,51\}$ $\{0,7,18,37,59\}$ $\{0,2,34,47\}$
$\{0,6,29,54\}$ $\{0,10,38,50\}$
- $v = 131$:
$\{0,4,16,44,64\}$ $\{0,5,42,63\}$ $\{0,10,33,57\}$ $\{0,22,30,61\}$ $\{0,1,3,53\}$ $\{0,6,19,51,65\}$ $\{0,7,41,56\}$
$\{0,11,29,54\}$ $\{0,26,35,62\}$ $\{0,17,55\}$
- $v = 143$:
$\{0,5,20,41,70\}$ $\{0,2,39,51\}$ $\{0,4,56,67\}$ $\{0,10,43,57\}$ $\{0,18,35,60\}$ $\{0,7,23,45,71\}$ $\{0,3,61,69\}$
$\{0,6,40,68\}$ $\{0,13,32,59\}$ $\{0,9,53\}$ $\{0,1,31,55\}$
- $v = 155$:
$\{0,6,21,56,76\}$ $\{0,13,41,59\}$ $\{0,1,26,58\}$ $\{0,3,45,67\}$ $\{0,5,43,73\}$ $\{0,8,24,60,77\}$ $\{0,23,33,72\}$
$\{0,2,29,63\}$ $\{0,4,44,75\}$ $\{0,9,74\}$ $\{0,11,48,62\}$ $\{0,47,54,66\}$
- $v = 161$:
$\{0,9,31,61,79\}$ $\{0,2,39,67\}$ $\{0,7,45,78\}$ $\{0,16,58,75\}$ $\{0,4,10\}$ $\{0,11,34,66,80\}$ $\{0,3,57,77\}$
$\{0,8,51,64\}$ $\{0,26,47,76\}$ $\{0,15,40\}$ $\{0,1,36,63\}$ $\{0,5,49,73\}$ $\{0,12,53,72\}$
- $v = 167$:
$\{0,6,25,57,83\}$ $\{0,2,44,67\}$ $\{0,5,52,76\}$ $\{0,36,64,74\}$ $\{0,14,30\}$ $\{0,7,27,60,82\}$ $\{0,3,46,59\}$
$\{0,9,48,79\}$ $\{0,37,49,66\}$ $\{0,15,78\}$ $\{0,1,41,62\}$ $\{0,4,54,72\}$ $\{0,11,45,80\}$ $\{0,8,81\}$
- $v = 173$:
$\{0,10,37,71,86\}$ $\{0,32,67,83\}$ $\{0,2,28,42\}$ $\{0,7,59,79\}$ $\{0,12,48\}$ $\{0,19,44,62,85\}$ $\{0,29,60,82\}$
$\{0,3,73,77\}$ $\{0,9,65,78\}$ $\{0,30,80\}$ $\{0,21,45,84\}$ $\{0,1,47,55\}$ $\{0,6,64,81\}$ $\{0,11,68\}$
$\{0,5,38\}$
- $v = 179$:
$\{0,8,28,60,89\}$ $\{0,2,51,67\}$ $\{0,10,66,85\}$ $\{0,18,82,87\}$ $\{0,1,38,74\}$ $\{0,9,30,63,88\}$ $\{0,3,44,86\}$
$\{0,15,55,68\}$ $\{0,22,48,72\}$ $\{0,12,47\}$ $\{0,27,31,70\}$ $\{0,6,77,84\}$ $\{0,17,62,76\}$ $\{0,23,34,80\}$
- $v = 185$:
$\{0,9,33,73,91\}$ $\{0,3,46,78\}$ $\{0,7,62,74\}$ $\{0,22,51,79\}$ $\{0,10,60\}$ $\{0,11,36,77,92\}$ $\{0,4,49,87\}$
$\{0,8,61,88\}$ $\{0,23,39,86\}$ $\{0,14,90\}$ $\{0,1,35,72\}$ $\{0,5,59,89\}$ $\{0,17,48,69\}$ $\{0,6,19\}$
$\{0,20,85\}$ $\{0,2,44,70\}$
- $v = 191$:
$\{0,6,27,67,95\}$ $\{0,3,56,75\}$ $\{0,11,60,90\}$ $\{0,39,77,85\}$ $\{0,9,91\}$ $\{0,7,29,70,94\}$ $\{0,4,59,84\}$
$\{0,14,62,78\}$ $\{0,43,76,88\}$ $\{0,13,31\}$ $\{0,1,51,74\}$ $\{0,10,57,93\}$ $\{0,26,58,92\}$ $\{0,44,81,86\}$
$\{0,15,35\}$ $\{0,2,54,71\}$
- $v = 197$:
$\{0,10,37,55,98\}$ $\{0,20,77,96\}$ $\{0,1,33,42\}$ $\{0,5,51,75\}$ $\{0,4,48\}$ $\{0,21,85,92\}$ $\{0,11,91\}$
$\{0,2,36,83\}$ $\{0,6,72,84\}$ $\{0,17,86\}$ $\{0,13,73,95\}$ $\{0,26,79,93\}$ $\{0,3,52,68\}$ $\{0,40,90\}$
$\{0,8,23,62,97\}$ $\{0,38,63,94\}$ $\{0,28,58,87\}$

- $v = 203:$

$\{0,8,31,67,100\}$ $\{0,20,80,97\}$ $\{0,4,49,90\}$ $\{0,9,57,85\}$ $\{0,40,96\}$ $\{0,14,82,98\}$ $\{0,42,89\}$
$\{0,5,70,83\}$ $\{0,11,55,74\}$ $\{0,34,61,87\}$ $\{0,24,88,95\}$ $\{0,1,39,51\}$ $\{0,6,81,99\}$ $\{0,3,46\}$
$\{0,10,35,72,101\}$ $\{0,21,79,94\}$ $\{0,2,32,54\}$

- $v = 209:$

$\{0,8,37,81,103\}$ $\{0,3,49,101\}$ $\{0,9,68,99\}$ $\{0,25,78,96\}$ $\{0,4,58,91\}$ $\{0,21,76\}$ $\{0,11,26\}$
$\{0,13,70,102\}$ $\{0,28,84,100\}$ $\{0,12,39\}$ $\{0,1,35,83\}$ $\{0,5,47,97\}$ $\{0,14,65,88\}$ $\{0,6,67\}$
$\{0,10,40,85,104\}$ $\{0,2,38,79\}$ $\{0,7,69,93\}$ $\{0,20,63,80\}$

- $v = 215:$

$\{0,7,31,78,106\}$ $\{0,3,44,103\}$ $\{0,27,50,89\}$ $\{0,45,64,94\}$ $\{0,4,87,95\}$ $\{0,29,97\}$ $\{0,20,60\}$
$\{0,12,38,105\}$ $\{0,46,61,104\}$ $\{0,6,92,102\}$ $\{0,1,33,81\}$ $\{0,21,84\}$ $\{0,13,66,82\}$ $\{0,5,57\}$
$\{0,9,34,85,107\}$ $\{0,2,37,79\}$ $\{0,11,65,101\}$ $\{0,18,74,88\}$ $\{0,17,72\}$

- $v = 221:$

$\{0,5,28,71,110\}$ $\{0,19,76,106\}$ $\{0,1,32,36\}$ $\{0,8,70,81\}$ $\{0,37,98\}$ $\{0,10,60,78\}$ $\{0,15,92\}$
$\{0,22,85,101\}$ $\{0,46,55,104\}$ $\{0,45,51,99\}$ $\{0,29,88,109\}$ $\{0,2,26,93\}$ $\{0,3,47,100\}$ $\{0,20,84\}$
$\{0,13,38,65,107\}$ $\{0,34,75,108\}$ $\{0,17,89,103\}$ $\{0,7,90,102\}$ $\{0,40,96\}$

- $v = 227:$

$\{0,8,31,61,112\}$ $\{0,16,60,105\}$ $\{0,1,36,65\}$ $\{0,5,59,102\}$ $\{0,7,46,108\}$ $\{0,21,98\}$ $\{0,19,88\}$
$\{0,2,40,73\}$ $\{0,20,92,106\}$ $\{0,18,67,109\}$ $\{0,11,94,111\}$ $\{0,6,84,93\}$ $\{0,3,58,99\}$ $\{0,15,90\}$
$\{0,10,34,66,113\}$ $\{0,12,80,107\}$ $\{0,25,82,110\}$ $\{0,4,52,74\}$ $\{0,13,50,76\}$

- $v = 233:$

$\{0,9,36,89,115\}$ $\{0,4,47,107\}$ $\{0,18,91,110\}$ $\{0,5,50,102\}$ $\{0,10,66\}$ $\{0,1,38,95\}$ $\{0,14,72\}$
$\{0,11,39,93,116\}$ $\{0,31,51,112\}$ $\{0,8,75,109\}$ $\{0,29,46,114\}$ $\{0,3,44,99\}$ $\{0,2,42,90\}$ $\{0,6,71\}$
$\{0,30,100,113\}$ $\{0,33,49,111\}$ $\{0,12,76,98\}$ $\{0,24,87,108\}$ $\{0,7,32\}$ $\{0,35,104\}$ $\{0,15,74\}$

- $v = 239:$

$\{0,8,33,88,118\}$ $\{0,4,68,115\}$ $\{0,42,58,102\}$ $\{0,14,81,113\}$ $\{0,2,39,93\}$ $\{0,11,116\}$ $\{0,9,71\}$
$\{0,10,36,92,119\}$ $\{0,45,104,117\}$ $\{0,49,66,114\}$ $\{0,1,35,87\}$ $\{0,20,43,96\}$ $\{0,22,73,97\}$ $\{0,15,78\}$
$\{0,31,101,108\}$ $\{0,6,100,112\}$ $\{0,21,40,90\}$ $\{0,28,46,107\}$ $\{0,3,41,98\}$ $\{0,29,103\}$ $\{0,5,89\}$

- $v = 245:$

$\{0,7,33,78,122\}$ $\{0,36,86,117\}$ $\{0,15,94,112\}$ $\{0,49,87,116\}$ $\{0,4,68,76\}$ $\{0,10,69,85\}$ $\{0,32,98\}$
$\{0,40,101,114\}$ $\{0,27,57,111\}$ $\{0,25,62,113\}$ $\{0,21,77,120\}$ $\{0,9,91,105\}$ $\{0,2,92,104\}$ $\{0,5,65\}$
$\{0,11,28,63,121\}$ $\{0,46,80,119\}$ $\{0,48,95,118\}$ $\{0,3,103,109\}$ $\{0,1,20,42\}$ $\{0,53,108\}$ $\{0,24,107\}$

- $v = 251:$

$\{0,8,31,71,124\}$ $\{0,13,102,123\}$ $\{0,12,97,113\}$ $\{0,9,61,118\}$ $\{0,4,81,88\}$ $\{0,11,69,94\}$ $\{0,33,80\}$
$\{0,17,104,122\}$ $\{0,14,106,121\}$ $\{0,28,74,76\}$ $\{0,19,55,117\}$ $\{0,1,67,112\}$ $\{0,3,54,103\}$ $\{0,39,95\}$
$\{0,10,34,75,125\}$ $\{0,26,86,108\}$ $\{0,20,90,119\}$ $\{0,6,78,120\}$ $\{0,32,59,96\}$ $\{0,30,68,73\}$ $\{0,35,79\}$

- $v = 257:$

$\{0,10,39,97,127\}$ $\{0,16,80,112\}$ $\{0,4,50,113\}$ $\{0,34,72,108\}$ $\{0,9,84,104\}$ $\{0,22,99\}$ $\{0,19,98\}$
$\{0,24,81,106\}$ $\{0,35,56,124\}$ $\{0,5,52,119\}$ $\{0,1,41,101\}$ $\{0,6,55,121\}$ $\{0,15,91\}$ $\{0,7,18\}$
$\{0,12,43,102,128\}$ $\{0,2,44,105\}$ $\{0,8,73,126\}$ $\{0,37,54,123\}$ $\{0,3,48,110\}$ $\{0,27,120\}$ $\{0,13,83\}$
$\{0,28,51,122\}$ $\{0,33,111,125\}$

- $v = 263:$

$\{0,9,35,98,130\}$ $\{0,44,50,125\}$ $\{0,33,51,112\}$ $\{0,7,106,122\}$ $\{0,12,123\}$ $\{0,4,46,104\}$ $\{0,19,113\}$
$\{0,52,119,129\}$ $\{0,54,74,127\}$ $\{0,57,72,128\}$ $\{0,22,88,109\}$ $\{0,8,78,126\}$ $\{0,1,37,97\}$ $\{0,13,30\}$
$\{0,11,38,102,131\}$ $\{0,2,41,103\}$ $\{0,14,90,124\}$ $\{0,25,84,107\}$ $\{0,3,43,108\}$ $\{0,49,117\}$ $\{0,5,85\}$
$\{0,24,69,116\}$ $\{0,28,83,114\}$

- $v = 269:$

$\{0,6,31,72,134\}$ $\{0,22,92,129\}$ $\{0,39,98,122\}$ $\{0,15,97,120\}$ $\{0,48,115\}$ $\{0,3,54,117\}$ $\{0,19,47\}$
$\{0,21,96,130\}$ $\{0,18,58,126\}$ $\{0,17,95,127\}$ $\{0,26,99,132\}$ $\{0,2,79,91\}$ $\{0,1,43,119\}$ $\{0,7,56\}$
$\{0,9,45,80,133\}$ $\{0,27,111,131\}$ $\{0,13,87,125\}$ $\{0,11,57,101\}$ $\{0,5,55,65\}$ $\{0,8,69,121\}$ $\{0,4,85\}$
$\{0,30,94,123\}$ $\{0,14,100,116\}$

- $v = 275:$

$\{0,10,36,81,136\}$ $\{0,15,110,131\}$ $\{0,34,67,127\}$ $\{0,6,49,135\}$ $\{0,1,41,78\}$ $\{0,8,76,120\}$ $\{0,38,94\}$
$\{0,30,65,134\}$ $\{0,22,111,128\}$ $\{0,25,97,121\}$ $\{0,4,54,118\}$ $\{0,7,70,99\}$ $\{0,13,74,88\}$ $\{0,47,105\}$
$\{0,12,39,85,137\}$ $\{0,28,119,130\}$ $\{0,23,113,132\}$ $\{0,2,82,124\}$ $\{0,9,66,117\}$ $\{0,31,79,84\}$ $\{0,59,62\}$
$\{0,32,115,133\}$ $\{0,16,103,123\}$

- $v = 281:$

$\{0,12,45,106,140\}$ $\{0,21,87,123\}$ $\{0,14,93,112\}$ $\{0,42,127,138\}$ $\{0,4,56,121\}$ $\{0,3,53,116\}$ $\{0,20,44\}$
$\{0,18,86,108\}$ $\{0,28,103,129\}$ $\{0,5,64,136\}$ $\{0,1,47,105\}$ $\{0,6,80,120\}$ $\{0,8,31\}$ $\{0,7,77\}$
$\{0,13,48,110,139\}$ $\{0,43,82,119\}$ $\{0,9,78,133\}$ $\{0,38,122,137\}$ $\{0,2,51,111\}$ $\{0,27,134\}$ $\{0,25,92\}$
$\{0,10,81,135\}$ $\{0,41,57,130\}$ $\{0,32,115,132\}$ $\{0,30,118\}$

- $v = 287:$

$\{0,10,37,108,142\}$ $\{0,33,95,124\}$ $\{0,22,66,119\}$ $\{0,59,83,140\}$ $\{0,3,45,118\}$ $\{0,7,96,116\}$ $\{0,19,77\}$
$\{0,49,125,136\}$ $\{0,63,80,141\}$ $\{0,25,94,117\}$ $\{0,2,43,113\}$ $\{0,4,52,127\}$ $\{0,1,39,107\}$ $\{0,13,60\}$
$\{0,12,40,112,143\}$ $\{0,35,121,137\}$ $\{0,8,82,138\}$ $\{0,54,84,139\}$ $\{0,32,133\}$ $\{0,14,134\}$ $\{0,5,51\}$
$\{0,21,88,114\}$ $\{0,36,126,135\}$ $\{0,64,79,129\}$ $\{0,6,110,128\}$

- $v = 293$:

$\{0,11,33,78,146\}$ $\{0,21,114,141\}$ $\{0,16,87,142\}$ $\{0,20,110,139\}$ $\{0,3,63,77\}$ $\{0,4,88,132\}$ $\{0,17,125\}$
$\{0,19,89,124\}$ $\{0,34,131,137\}$ $\{0,15,79,133\}$ $\{0,49,101,144\}$ $\{0,5,96,134\}$ $\{0,32,115\}$ $\{0,7,73\}$
$\{0,23,51,92,145\}$ $\{0,57,107,143\}$ $\{0,13,61,117\}$ $\{0,18,100,130\}$ $\{0,1,47,59\}$ $\{0,8,80,106\}$ $\{0,10,75\}$
$\{0,25,127,136\}$ $\{0,39,76,138\}$ $\{0,2,42,123\}$ $\{0,31,116,140\}$

- $v = 299$:

$\{0,10,41,81,148\}$ $\{0,25,124,146\}$ $\{0,11,79,143\}$ $\{0,33,109,136\}$ $\{0,6,55,98\}$ $\{0,3,53,91\}$ $\{0,51,135\}$
$\{0,18,122,130\}$ $\{0,29,106,129\}$ $\{0,9,70,142\}$ $\{0,16,117,141\}$ $\{0,1,46,94\}$ $\{0,2,58,115\}$ $\{0,39,126\}$
$\{0,12,44,86,149\}$ $\{0,17,119,140\}$ $\{0,37,73,120\}$ $\{0,34,131,145\}$ $\{0,4,69,82\}$ $\{0,7,66,96\}$ $\{0,52,114\}$
$\{0,28,118,144\}$ $\{0,35,95,110\}$ $\{0,5,85,139\}$ $\{0,20,128,147\}$

- $v = 305$:

$\{0,12,47,116,152\}$ $\{0,14,109,133\}$ $\{0,39,72,149\}$ $\{0,42,134,145\}$ $\{0,5,67,146\}$ $\{0,32,106\}$ $\{0,7,89\}$
$\{0,40,130,148\}$ $\{0,41,57,137\}$ $\{0,43,68,143\}$ $\{0,44,127,142\}$ $\{0,3,55,126\}$ $\{0,9,29\}$ $\{0,17,93\}$
$\{0,13,50,120,151\}$ $\{0,45,136,144\}$ $\{0,23,84,147\}$ $\{0,26,112,139\}$ $\{0,2,53,117\}$ $\{0,1,49,122\}$ $\{0,10,88\}$
$\{0,28,87,125\}$ $\{0,30,111,132\}$ $\{0,46,65,131\}$ $\{0,22,56,150\}$ $\{0,4,58,118\}$ $\{0,6,135\}$

- $v = 311$:

$\{0,11,39,118,154\}$ $\{0,54,112,129\}$ $\{0,1,41,117\}$ $\{0,37,131,147\}$ $\{0,65,90,152\}$ $\{0,19,93\}$ $\{0,5,60\}$
$\{0,34,105,136\}$ $\{0,72,84,141\}$ $\{0,63,95,151\}$ $\{0,15,104,126\}$ $\{0,64,91,150\}$ $\{0,3,49,130\}$ $\{0,21,98\}$
$\{0,13,42,122,155\}$ $\{0,20,119,145\}$ $\{0,23,70,137\}$ $\{0,68,92,153\}$ $\{0,48,66,148\}$ $\{0,50,101\}$ $\{0,9,53\}$
$\{0,35,108,138\}$ $\{0,52,135,149\}$ $\{0,6,139,146\}$ $\{0,38,134,144\}$ $\{0,4,124,132\}$ $\{0,2,45,123\}$

- $v = 317$:

$\{0,15,47,91,158\}$ $\{0,29,115,152\}$ $\{0,4,110,116\}$ $\{0,24,117,153\}$ $\{0,49,126,134\}$ $\{0,2,74,140\}$ $\{0,12,107\}$
$\{0,109,132,148\}$ $\{0,58,139,146\}$ $\{0,19,83,137\}$ $\{0,26,128,145\}$ $\{0,56,89,150\}$ $\{0,5,62,80\}$ $\{0,31,130\}$
$\{0,21,59,84,157\}$ $\{0,35,125,155\}$ $\{0,11,45,142\}$ $\{0,13,113,135\}$ $\{0,27,114,154\}$ $\{0,10,92\}$ $\{0,9,78\}$
$\{0,52,103,156\}$ $\{0,41,101,149\}$ $\{0,14,79,147\}$ $\{0,55,105,151\}$ $\{0,3,124,144\}$ $\{0,1,43,71\}$

- $v = 323$:

$\{0,13,41,81,160\}$ $\{0,26,137,156\}$ $\{0,16,94,141\}$ $\{0,25,132,153\}$ $\{0,12,110,155\}$ $\{0,5,61,134\}$ $\{0,55,148\}$
$\{0,34,82,152\}$ $\{0,8,77,109\}$ $\{0,3,60,99\}$ $\{0,10,74,154\}$ $\{0,20,124,151\}$ $\{0,2,54,92\}$ $\{0,58,121\}$
$\{0,15,44,86,161\}$ $\{0,11,83,150\}$ $\{0,1,51,88\}$ $\{0,36,95,138\}$ $\{0,17,122,140\}$ $\{0,4,89,120\}$ $\{0,66,142\}$
$\{0,24,127,157\}$ $\{0,22,135,158\}$ $\{0,6,97,106\}$ $\{0,35,84,149\}$ $\{0,14,126,159\}$ $\{0,7,53,115\}$

- $v = 329$:

$\{0,12,49,125,163\}$ $\{0,43,72,158\}$ $\{0,8,99,161\}$ $\{0,41,126,145\}$ $\{0,48,141,148\}$ $\{0,5,103\}$ $\{0,4,27\}$
$\{0,44,146,162\}$ $\{0,42,73,143\}$ $\{0,3,58,138\}$ $\{0,46,64,156\}$ $\{0,17,123,144\}$ $\{0,11,107\}$ $\{0,15,97\}$
$\{0,14,53,130,164\}$ $\{0,40,66,149\}$ $\{0,45,67,157\}$ $\{0,33,122,154\}$ $\{0,47,142,152\}$ $\{0,1,51,129\}$ $\{0,9,65\}$
$\{0,24,132,160\}$ $\{0,30,87,147\}$ $\{0,2,54,133\}$ $\{0,20,88,159\}$ $\{0,36,120,155\}$ $\{0,6,81,140\}$ $\{0,25,94\}$
$\{0,13,74,137\}$

- $v = 335$:

$\{0,12,41,128,166\}$ $\{0,5,50,152\}$ $\{0,67,95,164\}$ $\{0,68,144,159\}$ $\{0,56,110,163\}$ $\{0,3,52,134\}$ $\{0,17,103\}$
$\{0,34,113,146\}$ $\{0,39,71,148\}$ $\{0,9,120,133\}$ $\{0,37,143,151\}$ $\{0,58,117,139\}$ $\{0,1,43,127\}$ $\{0,6,24\}$
$\{0,14,44,132,167\}$ $\{0,61,92,162\}$ $\{0,36,47,155\}$ $\{0,64,80,158\}$ $\{0,60,150,160\}$ $\{0,2,48,137\}$ $\{0,25,129\}$
$\{0,40,63,161\}$ $\{0,21,83,157\}$ $\{0,4,142,149\}$ $\{0,66,93,165\}$ $\{0,26,122,141\}$ $\{0,57,130\}$ $\{0,51,156\}$
$\{0,65,85,140\}$

- $v = 341$:

$\{0,17,47,98,170\}$ $\{0,21,142,164\}$ $\{0,11,89,156\}$ $\{0,23,106,151\}$ $\{0,20,147,160\}$ $\{0,10,103\}$ $\{0,55,149\}$
$\{0,18,104,150\}$ $\{0,34,135,154\}$ $\{0,53,107\}$ $\{0,31,130,167\}$ $\{0,27,119,161\}$ $\{0,2,71,79\}$ $\{0,62,152\}$
$\{0,25,137,163\}$ $\{0,32,129,165\}$ $\{0,3,105,111\}$ $\{0,41,116,155\}$ $\{0,40,131,166\}$ $\{0,5,73,87\}$ $\{0,16,74\}$
$\{0,28,52,85,169\}$ $\{0,70,146,158\}$ $\{0,7,66,122\}$ $\{0,44,124,162\}$ $\{0,43,139,168\}$ $\{0,1,61,65\}$ $\{0,50,113\}$
$\{0,9,109,157\}$ $\{0,15,110,159\}$

- $v = 347$:

$\{0,14,61,91,172\}$ $\{0,20,142,165\}$ $\{0,7,75,163\}$ $\{0,17,143,170\}$ $\{0,5,103,141\}$ $\{0,8,71,112\}$ $\{0,12,132\}$
$\{0,28,118,147\}$ $\{0,15,154,167\}$ $\{0,2,58,123\}$ $\{0,44,83,150\}$ $\{0,1,53,102\}$ $\{0,50,144\}$ $\{0,35,134\}$
$\{0,16,64,95,173\}$ $\{0,73,113,155\}$ $\{0,18,87,133\}$ $\{0,34,127,164\}$ $\{0,4,59,135\}$ $\{0,3,57,100\}$ $\{0,60,168\}$
$\{0,19,70,159\}$ $\{0,36,128,161\}$ $\{0,11,96,162\}$ $\{0,32,148,169\}$ $\{0,10,72,117\}$ $\{0,6,80,166\}$ $\{0,26,110\}$
$\{0,22,146,171\}$ $\{0,24,129,138\}$

- $v = 353$:

$\{0,14,51,133,175\}$ $\{0,18,107,169\}$ $\{0,6,69,164\}$ $\{0,20,120,168\}$ $\{0,32,110,145\}$ $\{0,60,140\}$ $\{0,7,75\}$
$\{0,36,132,166\}$ $\{0,22,106,172\}$ $\{0,19,90,162\}$ $\{0,1,53,139\}$ $\{0,25,104,174\}$ $\{0,2,58,156\}$ $\{0,3,13\}$
$\{0,23,117,157\}$ $\{0,41,118,167\}$ $\{0,31,55,147\}$ $\{0,45,74,173\}$ $\{0,33,135,144\}$ $\{0,15,108\}$ $\{0,8,67\}$
$\{0,16,54,137,176\}$ $\{0,46,127,171\}$ $\{0,43,64,155\}$ $\{0,47,73,170\}$ $\{0,30,131,159\}$ $\{0,4,61,146\}$ $\{0,17,105\}$
$\{0,50,153,165\}$ $\{0,27,136,141\}$ $\{0,11,76,163\}$

- $v = 359$:

$\{0,13,43,138,178\}$ $\{0,41,122,158\}$ $\{0,1,45,137\}$ $\{0,63,87,176\}$ $\{0,73,101,177\}$ $\{0,8,60,168\}$ $\{0,4,159\}$
$\{0,15,46,142,179\}$ $\{0,32,150,162\}$ $\{0,5,115,166\}$ $\{0,75,131,153\}$ $\{0,26,53,152\}$ $\{0,14,62\}$ $\{0,50,140\}$
$\{0,20,144,169\}$ $\{0,68,106,175\}$ $\{0,39,93,148\}$ $\{0,66,85,171\}$ $\{0,58,79,170\}$ $\{0,7,64,146\}$ $\{0,3,119\}$
$\{0,42,71,174\}$ $\{0,33,121,156\}$ $\{0,6,134,151\}$ $\{0,61,72,172\}$ $\{0,59,157,173\}$ $\{0,2,49,143\}$ $\{0,23,97\}$
$\{0,65,83,167\}$ $\{0,34,154,163\}$ $\{0,70,80,147\}$

- $v = 365$:

$\{0,29,93,151,182\}$ $\{0,37,134,167\}$ $\{0,43,94,174\}$ $\{0,11,114,169\}$ $\{0,15,150,172\}$ $\{0,38,140\}$ $\{0,3,108\}$
$\{0,14,152,170\}$ $\{0,17,96,117\}$ $\{0,25,85,166\}$ $\{0,49,111,178\}$ $\{0,34,149,162\}$ $\{0,2,74,84\}$ $\{0,5,104\}$
$\{0,20,143,175\}$ $\{0,52,118,179\}$ $\{0,48,98,173\}$ $\{0,9,142,168\}$ $\{0,30,154,177\}$ $\{0,16,164\}$ $\{0,7,139\}$
$\{0,36,90,137,181\}$ $\{0,78,119,161\}$ $\{0,71,77,163\}$ $\{0,56,113,176\}$ $\{0,68,121,180\}$ $\{0,1,46,70\}$ $\{0,4,110\}$
$\{0,76,88,116\}$ $\{0,35,144,171\}$ $\{0,65,73,160\}$ $\{0,39,146,165\}$

- $v = 371$:

$\{0,14,61,94,184\}$ $\{0,30,147,171\}$ $\{0,8,67,132\}$ $\{0,35,88,173\}$ $\{0,12,139,146\}$ $\{0,1,55,96\}$ $\{0,71,176\}$
$\{0,16,64,98,185\}$ $\{0,42,116,162\}$ $\{0,32,83,174\}$ $\{0,11,133,177\}$ $\{0,15,92,167\}$ $\{0,3,63,100\}$ $\{0,10,113\}$
$\{0,26,144,175\}$ $\{0,22,151,179\}$ $\{0,2,58,101\}$ $\{0,36,112,181\}$ $\{0,20,126,135\}$ $\{0,45,154\}$ $\{0,40,102\}$
$\{0,19,159,172\}$ $\{0,38,119,168\}$ $\{0,4,108,114\}$ $\{0,39,107,164\}$ $\{0,27,158,183\}$ $\{0,5,78,148\}$ $\{0,66,150\}$
$\{0,21,93,182\}$ $\{0,17,128,180\}$ $\{0,29,79,165\}$ $\{0,23,160,178\}$

- $v = 377$:

$\{0,16,54,141,187\}$ $\{0,7,120,179\}$ $\{0,50,80,182\}$ $\{0,52,164,178\}$ $\{0,49,158,177\}$ $\{0,2,60,157\}$ $\{0,20,83\}$
$\{0,18,57,145,188\}$ $\{0,21,116,183\}$ $\{0,42,77,180\}$ $\{0,45,74,181\}$ $\{0,10,100,161\}$ $\{0,24,134\}$ $\{0,12,25\}$
$\{0,22,140,174\}$ $\{0,40,139,175\}$ $\{0,27,92,186\}$ $\{0,41,142,165\}$ $\{0,15,108,184\}$ $\{0,9,81,163\}$ $\{0,8,78\}$
$\{0,44,130,167\}$ $\{0,47,143,176\}$ $\{0,26,111\}$ $\{0,3,69,153\}$ $\{0,4,75,148\}$ $\{0,1,56,147\}$ $\{0,5,119\}$
$\{0,53,64,168\}$ $\{0,32,121,149\}$ $\{0,6,68,166\}$ $\{0,48,79,185\}$ $\{0,51,156,173\}$

- $v = 383$:

$\{0,14,45,148,190\}$ $\{0,62,168,187\}$ $\{0,44,81,167\}$ $\{0,80,97,169\}$ $\{0,28,82,172\}$ $\{0,12,178\}$ $\{0,6,155\}$
$\{0,34,118,173\}$ $\{0,58,165,180\}$ $\{0,68,91,184\}$ $\{0,78,113,188\}$ $\{0,2,51,153\}$ $\{0,59,128\}$ $\{0,13,70\}$
$\{0,16,48,152,191\}$ $\{0,60,159,181\}$ $\{0,43,73,185\}$ $\{0,40,76,164\}$ $\{0,26,53,158\}$ $\{0,3,64,160\}$ $\{0,18,101\}$
$\{0,71,95,182\}$ $\{0,66,174,183\}$ $\{0,65,94,179\}$ $\{0,63,74,189\}$ $\{0,4,56,154\}$ $\{0,1,47,147\}$ $\{0,5,135\}$
$\{0,33,162,170\}$ $\{0,21,141,161\}$ $\{0,7,138,163\}$ $\{0,38,79,171\}$ $\{0,67,77,186\}$ $\{0,50,177\}$

- $v = 389$:

$\{0,13,78,110,194\}$ $\{0,47,135,187\}$ $\{0,35,75,179\}$ $\{0,21,143,182\}$ $\{0,34,145,176\}$ $\{0,15,114\}$ $\{0,12,80\}$
$\{0,29,86,123,193\}$ $\{0,33,150,186\}$ $\{0,3,51,149\}$ $\{0,56,120,192\}$ $\{0,38,141,185\}$ $\{0,2,81,95\}$ $\{0,76,166\}$
$\{0,20,151,178\}$ $\{0,58,124,191\}$ $\{0,54,82,167\}$ $\{0,50,105,188\}$ $\{0,25,162,184\}$ $\{0,46,152\}$ $\{0,42,168\}$
$\{0,26,160,183\}$ $\{0,43,170,175\}$ $\{0,7,155,172\}$ $\{0,45,154,173\}$ $\{0,59,112,189\}$ $\{0,1,63,74\}$ $\{0,49,174\}$
$\{0,18,89,119\}$ $\{0,61,121,190\}$ $\{0,6,169,177\}$ $\{0,41,156,180\}$ $\{0,10,102,118\}$ $\{0,4,91,100\}$

- $v = 395$:

$\{0,14,61,94,196\}$ $\{0,19,167,190\}$ $\{0,32,85,176\}$ $\{0,24,153,179\}$ $\{0,28,166,191\}$ $\{0,3,71,161\}$ $\{0,5,109\}$
$\{0,16,64,98,197\}$ $\{0,21,157,193\}$ $\{0,37,93,188\}$ $\{0,11,107,194\}$ $\{0,22,156,187\}$ $\{0,42,125\}$ $\{0,39,178\}$
$\{0,52,140,195\}$ $\{0,13,119,137\}$ $\{0,2,69,123\}$ $\{0,43,92,173\}$ $\{0,7,120,192\}$ $\{0,4,74,118\}$ $\{0,60,170\}$
$\{0,27,128,168\}$ $\{0,50,147,177\}$ $\{0,12,75,186\}$ $\{0,41,103,149\}$ $\{0,17,159,169\}$ $\{0,1,66,117\}$ $\{0,6,132\}$
$\{0,9,131,189\}$ $\{0,15,115,160\}$ $\{0,20,79,184\}$ $\{0,29,86,175\}$ $\{0,38,73,150\}$ $\{0,8,84,162\}$

- $v = 401$:

$\{0,20,56,151,199\}$ $\{0,26,129,160\}$ $\{0,46,81,196\}$ $\{0,25,137,166\}$ $\{0,34,154,187\}$ $\{0,23,128\}$ $\{0,5,77\}$
$\{0,22,60,157,200\}$ $\{0,12,106,194\}$ $\{0,53,69,186\}$ $\{0,21,76,189\}$ $\{0,49,86,195\}$ $\{0,15,136\}$ $\{0,7,75\}$
$\{0,13,96,188\}$ $\{0,52,142,184\}$ $\{0,50,82,198\}$ $\{0,45,130,169\}$ $\{0,14,87,197\}$ $\{0,6,71,171\}$ $\{0,9,127\}$
$\{0,51,79,190\}$ $\{0,54,145,192\}$ $\{0,2,61,163\}$ $\{0,41,155,185\}$ $\{0,8,74,172\}$ $\{0,18,180\}$ $\{0,3,67\}$
$\{0,11,119,181\}$ $\{0,19,123,193\}$ $\{0,10,99,177\}$ $\{0,44,84,191\}$ $\{0,24,149,176\}$ $\{0,1,58,159\}$ $\{0,4,126\}$
$\{0,17,80,173\}$

- $v = 407$:

$\{0,15,47,158,202\}$ $\{0,76,171,183\}$ $\{0,11,66,179\}$ $\{0,18,141,169\}$ $\{0,14,147,181\}$ $\{0,9,36\}$ $\{0,4,142\}$
$\{0,17,50,162,203\}$ $\{0,65,182,192\}$ $\{0,37,67,185\}$ $\{0,24,149,174\}$ $\{0,43,144,189\}$ $\{0,1,49,157\}$ $\{0,21,106\}$
$\{0,77,100,196\}$ $\{0,78,116,200\}$ $\{0,2,53,163\}$ $\{0,70,92,194\}$ $\{0,64,132,195\}$ $\{0,3,57,191\}$ $\{0,52,114\}$
$\{0,81,121,201\}$ $\{0,16,105,193\}$ $\{0,7,172,180\}$ $\{0,69,82,197\}$ $\{0,42,139,178\}$ $\{0,5,61,159\}$ $\{0,59,199\}$
$\{0,99,130,190\}$ $\{0,90,109,184\}$ $\{0,6,135,170\}$ $\{0,46,72,198\}$ $\{0,83,103,176\}$ $\{0,71,175\}$ $\{0,74,160\}$
$\{0,79,137,166\}$

- $v = 425$:

$\{0,24,58,161,211\}$ $\{0,18,129,199\}$ $\{0,52,89,208\}$ $\{0,55,154,195\}$ $\{0,12,128,201\}$ $\{0,2,64,170\}$ $\{0,7,28\}$
$\{0,26,61,165,212\}$ $\{0,17,100,190\}$ $\{0,3,66,174\}$ $\{0,1,60,167\}$ $\{0,15,130,206\}$ $\{0,22,136\}$ $\{0,8,134\}$
$\{0,23,118,205\}$ $\{0,40,160,192\}$ $\{0,11,79,188\}$ $\{0,46,184,203\}$ $\{0,16,101,194\}$ $\{0,14,135\}$ $\{0,10,88\}$
$\{0,27,124,210\}$ $\{0,54,148,204\}$ $\{0,51,84,209\}$ $\{0,39,185,198\}$ $\{0,48,144,197\}$ $\{0,4,69,179\}$ $\{0,20,132\}$
$\{0,57,82,180\}$ $\{0,45,162,200\}$ $\{0,5,77,169\}$ $\{0,49,91,196\}$ $\{0,43,145,176\}$ $\{0,6,81\}$ $\{0,67,141\}$
$\{0,9,131,202\}$ $\{0,30,143,172\}$ $\{0,44,80,207\}$

- $v = 431$:

$\{0,23,49,203,215\}$ $\{0,56,116,175\}$ $\{0,5,183,201\}$ $\{0,94,113,191\}$ $\{0,81,114,204\}$ $\{0,4,61,167\}$ $\{0,6,171\}$
$\{0,41,75,211,214\}$ $\{0,62,133,200\}$ $\{0,48,83,209\}$ $\{0,91,108,195\}$ $\{0,70,185,205\}$ $\{0,1,51,169\}$ $\{0,8,164\}$
$\{0,74,159,199\}$ $\{0,77,105,207\}$ $\{0,72,86,213\}$ $\{0,37,181,194\}$ $\{0,84,148,179\}$ $\{0,76,197\}$ $\{0,9,143\}$
$\{0,82,103,210\}$ $\{0,65,158,182\}$ $\{0,69,96,206\}$ $\{0,44,142,184\}$ $\{0,80,112,212\}$ $\{0,2,54,174\}$ $\{0,15,58\}$
$\{0,89,155,177\}$ $\{0,16,147,202\}$ $\{0,47,92,193\}$ $\{0,25,176,187\}$ $\{0,36,160,189\}$ $\{0,68,190\}$ $\{0,10,73\}$
$\{0,38,149,188\}$ $\{0,79,109,208\}$ $\{0,46,53,198\}$

- $v = 449$:

$\{0,28,63,175,224\}$ $\{0,18,106,189\}$ $\{0,56,90,219\}$ $\{0,42,173,206\}$ $\{0,9,105,191\}$ $\{0,31,140\}$ $\{0,8,144\}$
$\{0,30,67,178,223\}$ $\{0,36,174,198\}$ $\{0,1,66,180\}$ $\{0,40,159,186\}$ $\{0,11,95,221\}$ $\{0,54,92,220\}$ $\{0,10,25\}$
$\{0,13,152,216\}$ $\{0,52,100,197\}$ $\{0,3,72,187\}$ $\{0,53,157,208\}$ $\{0,2,70,183\}$ $\{0,14,151\}$ $\{0,12,89\}$
$\{0,29,132,214\}$ $\{0,43,170,211\}$ $\{0,7,130,209\}$ $\{0,44,194,213\}$ $\{0,4,75,192\}$ $\{0,59,76,177\}$ $\{0,20,154\}$
$\{0,16,110,217\}$ $\{0,22,121,212\}$ $\{0,5,78,200\}$ $\{0,61,142,204\}$ $\{0,46,85,218\}$ $\{0,21,141\}$ $\{0,26,124\}$
$\{0,47,149,207\}$ $\{0,57,165,215\}$ $\{0,6,80,205\}$ $\{0,60,153,176\}$ $\{0,55,87,222\}$

B Small $(4t, 4, \{3, 5\}, 1)$-PDFs for $t \equiv 0 \pmod 6$

Here we list all the base blocks of a $(4t, 4, \{3, 5\}, 1)$-PDF for $t \equiv 0 \pmod 6$ and $6 \le t \le 54$.

- $t = 6$: $\{0, 1, 4, 9, 11\}$
- $t = 12$: $\{0, 1, 3, 9, 20\}$ $\{0, 4, 18\}$ $\{0, 5, 21\}$ $\{0, 7, 22\}$ $\{0, 10, 23\}$
- $t = 18$:

$\{0, 1, 3, 7, 24\}$ $\{0, 8, 35\}$ $\{0, 10, 29\}$ $\{0, 12, 32\}$ $\{0, 14, 30\}$ $\{0, 5, 31\}$ $\{0, 9, 34\}$
$\{0, 11, 33\}$ $\{0, 13, 28\}$

- $t = 24$:

$\{0, 1, 7, 20, 41\}$ $\{0, 4, 30\}$ $\{0, 9, 47\}$ $\{0, 12, 44\}$ $\{0, 16, 39\}$ $\{0, 2, 17\}$ $\{0, 5, 36\}$
$\{0, 10, 43\}$ $\{0, 14, 42\}$ $\{0, 18, 45\}$ $\{0, 3, 25\}$ $\{0, 8, 37\}$ $\{0, 11, 46\}$

- $t = 30$:

$\{0, 1, 3, 32, 56\}$ $\{0, 13, 20, 41, 59\}$ $\{0, 23, 50\}$ $\{0, 8, 52\}$ $\{0, 12, 48\}$ $\{0, 22, 47\}$ $\{0, 15, 34\}$
$\{0, 4, 9, 42, 58\}$ $\{0, 6, 17, 43, 57\}$ $\{0, 10, 45\}$

- $t = 36$:

$\{0, 1, 3, 38, 68\}$ $\{0, 4, 9, 43, 70\}$ $\{0, 12, 64\}$ $\{0, 19, 60\}$ $\{0, 22, 55\}$ $\{0, 20, 51\}$ $\{0, 8, 62\}$
$\{0, 14, 25, 42, 71\}$ $\{0, 6, 13, 53, 69\}$ $\{0, 15, 59\}$ $\{0, 23, 49\}$ $\{0, 10, 58\}$ $\{0, 18, 50\}$ $\{0, 21, 45\}$

- $t = 42$:

$\{0, 1, 3, 44, 77\}$ $\{0, 10, 21, 58, 81\}$ $\{0, 13, 51, 67, 82\}$ $\{0, 4, 9, 49, 79\}$ $\{0, 8, 65\}$ $\{0, 26, 73\}$ $\{0, 14, 66\}$
$\{0, 12, 34, 62, 80\}$ $\{0, 20, 27, 59, 83\}$ $\{0, 6, 25, 61, 78\}$ $\{0, 29, 64\}$

- $t = 48$:

$\{0, 1, 3, 50, 89\}$ $\{0, 8, 18, 71, 93\}$ $\{0, 17, 33, 74, 95\}$ $\{0, 23, 79\}$ $\{0, 28, 65\}$ $\{0, 14, 81\}$ $\{0, 25, 68\}$
$\{0, 4, 9, 55, 91\}$ $\{0, 11, 30, 70, 94\}$ $\{0, 12, 38, 72, 92\}$ $\{0, 29, 73\}$ $\{0, 31, 66\}$ $\{0, 15, 76\}$ $\{0, 27, 69\}$
$\{0, 6, 13, 58, 90\}$

- $t = 54$:

$\{0, 1, 3, 56, 99\}$ $\{0, 26, 67, 89, 101\}$ $\{0, 24, 72, 92, 103\}$ $\{0, 18, 64\}$ $\{0, 32, 81\}$ $\{0, 10, 94\}$ $\{0, 21, 80\}$
$\{0, 4, 9, 66, 104\}$ $\{0, 16, 58, 87, 102\}$ $\{0, 23, 50, 83, 97\}$ $\{0, 36, 76\}$ $\{0, 39, 91\}$ $\{0, 13, 106\}$ $\{0, 28, 105\}$
$\{0, 19, 25, 70, 107\}$ $\{0, 8, 69\}$ $\{0, 7, 85\}$ $\{0, 17, 90\}$ $\{0, 30, 65\}$

C Small $(4t, 4, \{3, 5\}, 1)$-PDFs for $t \equiv 2 \pmod 6$

Here we list all the base blocks of a $(4t, 4, \{3, 5\}, 1)$-PDF for $t \equiv 2 \pmod 6$ and $14 \le t \le 110$.

- $t = 14$: $\{0, 1, 9, 22, 25\}$ $\{0, 4, 10, 15, 27\}$ $\{0, 2, 20\}$ $\{0, 7, 26\}$
- $t = 20$:

$\{0, 1, 3, 7, 32\}$ $\{0, 5, 13, 28, 39\}$ $\{0, 9, 30\}$ $\{0, 12, 36\}$ $\{0, 16, 38\}$ $\{0, 17, 35\}$ $\{0, 10, 37\}$
$\{0, 14, 33\}$

- $t = 26$:

$\{0, 1, 4, 25, 38\}$ $\{0, 2, 7, 29, 43\}$ $\{0, 8, 47\}$ $\{0, 12, 44\}$ $\{0, 17, 50\}$ $\{0, 19, 42\}$ $\{0, 10, 45\}$
$\{0, 16, 46\}$ $\{0, 18, 49\}$ $\{0, 20, 48\}$ $\{0, 6, 15\}$ $\{0, 11, 51\}$

- $t = 32$:

$\{0, 1, 3, 34, 59\}$ $\{0, 9, 20, 46, 63\}$ $\{0, 15, 42, 50, 55\}$ $\{0, 14, 21, 44, 62\}$ $\{0, 24, 53\}$ $\{0, 16, 52\}$ $\{0, 22, 60\}$
$\{0, 4, 10, 49, 61\}$ $\{0, 19, 47\}$

- $t = 38$:

$\{0, 1, 3, 40, 71\}$ $\{0, 8, 19, 65, 75\}$ $\{0, 6, 13, 55, 72\}$ $\{0, 22, 54\}$ $\{0, 25, 52\}$ $\{0, 18, 61\}$ $\{0, 24, 50\}$
$\{0, 4, 9, 45, 73\}$ $\{0, 16, 30, 51, 74\}$ $\{0, 29, 63\}$ $\{0, 15, 62\}$ $\{0, 12, 60\}$ $\{0, 20, 53\}$

- $t = 44$:

$\{0, 1, 3, 46, 83\}$ $\{0, 20, 32, 60, 86\}$ $\{0, 16, 78\}$ $\{0, 22, 51\}$ $\{0, 25, 63\}$ $\{0, 19, 77\}$ $\{0, 7, 56\}$
$\{0, 5, 9, 74, 84\}$ $\{0, 6, 14, 48, 87\}$ $\{0, 23, 64\}$ $\{0, 27, 57\}$ $\{0, 11, 61\}$ $\{0, 21, 76\}$ $\{0, 24, 71\}$
$\{0, 15, 33, 68, 85\}$ $\{0, 31, 67\}$ $\{0, 13, 72\}$

- $t = 50$:

$\{0, 1, 3, 49, 92\}$ $\{0, 15, 26, 70, 97\}$ $\{0, 22, 67, 84, 98\}$ $\{0, 18, 37, 75, 96\}$ $\{0, 12, 81\}$ $\{0, 29, 68\}$ $\{0, 8, 73\}$
$\{0, 4, 9, 60, 94\}$ $\{0, 6, 58, 86, 93\}$ $\{0, 20, 33, 74, 99\}$ $\{0, 23, 53, 63, 95\}$ $\{0, 16, 77\}$ $\{0, 36, 83\}$ $\{0, 24, 88\}$

- $t = 56$:

$\{0, 1, 3, 58, 101\}$ $\{0, 16, 64, 92, 111\}$ $\{0, 22, 73, 93, 110\}$ $\{0, 27, 109\}$ $\{0, 33, 77\}$ $\{0, 21, 106\}$
$\{0, 4, 9, 63, 103\}$ $\{0, 6, 13, 66, 102\}$ $\{0, 25, 75, 90, 104\}$ $\{0, 11, 83\}$ $\{0, 31, 80\}$ $\{0, 34, 108\}$
$\{0, 8, 18, 70, 105\}$ $\{0, 26, 38, 68, 107\}$ $\{0, 23, 84\}$ $\{0, 32, 78\}$ $\{0, 41, 86\}$ $\{0, 24, 91\}$

- $t = 62$:

$\{0,1,3,64,113\}$	$\{0,25,47,100,121\}$	$\{0,26,43,83,116\}$	$\{0,15,95\}$	$\{0,29,122\}$	$\{0,35,86\}$
$\{0,4,9,69,115\}$	$\{0,6,13,72,114\}$	$\{0,27,81,105,119\}$	$\{0,16,118\}$	$\{0,30,85\}$	$\{0,37,104\}$
$\{0,8,18,76,117\}$	$\{0,20,39,91,123\}$	$\{0,23,120\}$	$\{0,31,87\}$	$\{0,44,89\}$	$\{0,11,88\}$
$\{0,28,107\}$	$\{0,34,70\}$	$\{0,48,98\}$	$\{0,12,94\}$		

- $t = 68$:

$\{0,1,3,70,125\}$	$\{0,29,56,91,133\}$	$\{0,23,38,82,130\}$	$\{0,11,100,116,128\}$	$\{0,40,113\}$	$\{0,13,112\}$
$\{0,46,111\}$	$\{0,4,9,75,127\}$	$\{0,34,51,87,132\}$	$\{0,14,74,115,135\}$	$\{0,37,95\}$	$\{0,50,114\}$
$\{0,10,57,96,129\}$	$\{0,25,88,109,131\}$	$\{0,6,32,108,126\}$	$\{0,8,93\}$	$\{0,19,97\}$	$\{0,7,90\}$
$\{0,31,80,110,134\}$					

- $t = 74$:

$\{0,1,3,76,130\}$	$\{0,26,86,120,147\}$	$\{0,14,38,106,136\}$	$\{0,23,114\}$	$\{0,42,138\}$	$\{0,37,100\}$
$\{0,28,50,90,143\}$	$\{0,6,13,84,131\}$	$\{0,35,56,105,144\}$	$\{0,8,119\}$	$\{0,25,112\}$	$\{0,43,101\}$
$\{0,10,46,113,145\}$	$\{0,16,57,123,140\}$	$\{0,11,29,80,139\}$	$\{0,12,116\}$	$\{0,31,126\}$	$\{0,48,133\}$
$\{0,33,52,97,141\}$	$\{0,4,9,81,146\}$	$\{0,55,134\}$	$\{0,15,132\}$	$\{0,20,102\}$	

- $t = 80$:

$\{0,1,3,82,146\}$	$\{0,33,92,136,152\}$	$\{0,4,9,87,148\}$	$\{0,10,75\}$	$\{0,23,131\}$	$\{0,43,158\}$
$\{0,34,110,135,155\}$	$\{0,6,13,90,147\}$	$\{0,35,54,107,153\}$	$\{0,14,109\}$	$\{0,24,149\}$	$\{0,48,114\}$
$\{0,28,40,140,151\}$	$\{0,36,58,105,156\}$	$\{0,17,150\}$	$\{0,31,127\}$	$\{0,49,122\}$	$\{0,18,124\}$
$\{0,29,68,142,157\}$	$\{0,37,63,104,154\}$	$\{0,38,94\}$	$\{0,52,138\}$	$\{0,21,137\}$	$\{0,42,130\}$
$\{0,30,62,132,159\}$	$\{0,55,126\}$	$\{0,8,93\}$			

- $t = 86$:

$\{0,1,3,88,152\}$	$\{0,16,83,94,163\}$	$\{0,34,106,141,162\}$	$\{0,36,61,111,170\}$	$\{0,48,119\}$	$\{0,27,157\}$
$\{0,26,45,126,169\}$	$\{0,40,98,137,161\}$	$\{0,6,13,114,166\}$	$\{0,38,120,140,171\}$	$\{0,55,132\}$	$\{0,14,136\}$
$\{0,30,92,145,168\}$	$\{0,8,65,112,156\}$	$\{0,37,110,142,164\}$	$\{0,18,135\}$	$\{0,60,139\}$	$\{0,15,118\}$
$\{0,12,29,125,158\}$	$\{0,42,70,116,165\}$	$\{0,4,9,93,159\}$	$\{0,68,167\}$	$\{0,10,154\}$	$\{0,41,131\}$

- $t = 92$:

$\{0,1,3,94,158\}$	$\{0,38,65,120,177\}$	$\{0,46,121,152,174\}$	$\{0,32,182\}$	$\{0,68,170\}$	$\{0,10,76\}$
$\{0,19,39,154,175\}$	$\{0,15,101,137,179\}$	$\{0,6,13,116,172\}$	$\{0,50,146\}$	$\{0,40,117\}$	$\{0,72,151\}$
$\{0,8,127,138,171\}$	$\{0,12,35,109,180\}$	$\{0,4,9,99,169\}$	$\{0,48,173\}$	$\{0,85,143\}$	$\{0,17,140\}$
$\{0,14,98,114,161\}$	$\{0,41,124,149,183\}$	$\{0,28,52,132,181\}$	$\{0,87,148\}$	$\{0,18,144\}$	$\{0,54,167\}$
$\{0,37,67,118,178\}$	$\{0,45,107,133,176\}$	$\{0,89,162\}$	$\{0,29,134\}$		

- $t = 98$:

$\{0,1,23,100,156\}$	$\{0,40,113,157,194\}$	$\{0,45,109,161,187\}$	$\{0,2,67,139,170\}$	$\{0,11,107\}$	$\{0,32,112\}$
$\{0,33,54,158,183\}$	$\{0,41,68,127,190\}$	$\{0,6,75,165,181\}$	$\{0,42,124,160,195\}$	$\{0,9,152\}$	$\{0,43,166\}$
$\{0,8,105,179,193\}$	$\{0,46,70,131,191\}$	$\{0,47,62,177,182\}$	$\{0,13,132,151,180\}$	$\{0,51,162\}$	$\{0,76,178\}$
$\{0,38,50,164,184\}$	$\{0,49,79,136,189\}$	$\{0,18,84,101,192\}$	$\{0,39,94,128,186\}$	$\{0,10,173\}$	$\{0,93,188\}$
$\{0,28,169,172,176\}$					

- $t = 104$:

$\{0,1,39,107,184\}$	$\{0,49,86,136,202\}$	$\{0,20,103,194,200\}$	$\{0,2,123,142,198\}$	$\{0,22,161\}$	$\{0,16,154\}$
$\{0,17,122,146,204\}$	$\{0,5,113,117,182\}$	$\{0,40,81,128,206\}$	$\{0,51,144,169,203\}$	$\{0,3,151\}$	$\{0,29,188\}$
$\{0,12,96,111,190\}$	$\{0,43,115,175,205\}$	$\{0,18,119,127,189\}$	$\{0,33,143,170,191\}$	$\{0,7,192\}$	$\{0,53,167\}$
$\{0,42,73,134,197\}$	$\{0,46,98,126,193\}$	$\{0,9,181\}$	$\{0,54,195\}$	$\{0,11,179\}$	$\{0,55,157\}$
$\{0,44,76,133,207\}$	$\{0,26,176,186,199\}$	$\{0,45,130,165,201\}$	$\{0,14,149\}$	$\{0,64,164\}$	

- $t = 110$:

$\{0,1,23,112,185\}$	$\{0,52,152,173,212\}$	$\{0,2,20,127,189\}$	$\{0,55,101,157,200\}$	$\{0,67,206\}$	$\{0,9,126\}$
$\{0,3,61,196,208\}$	$\{0,28,158,182,207\}$	$\{0,68,201\}$	$\{0,10,188\}$	$\{0,76,195\}$	$\{0,11,120\}$
$\{0,45,79,143,215\}$	$\{0,54,95,198,213\}$	$\{0,47,84,141,218\}$	$\{0,75,156\}$	$\{0,13,177\}$	$\{0,86,174\}$
$\{0,48,140,172,214\}$	$\{0,78,191\}$	$\{0,16,197\}$	$\{0,87,138\}$	$\{0,4,108\}$	$\{0,17,123\}$
$\{0,40,137,168,203\}$	$\{0,44,82,175,211\}$	$\{0,91,150\}$	$\{0,5,204\}$	$\{0,30,146\}$	$\{0,83,153\}$
$\{0,26,53,122,202\}$	$\{0,6,216\}$	$\{0,50,192\}$	$\{0,85,148\}$	$\{0,7,190\}$	$\{0,71,186\}$
$\{0,29,161,180,194\}$	$\{0,90,155\}$	$\{0,8,217\}$	$\{0,105,219\}$		

D Partition the set $[1, 2u+1] \setminus \{(u-14)/2\}$ into pairs

Here for each $u \in [36, 131]$ and $u \equiv 2 \pmod 4$, we give a partition of the set $[1, 2u+1] \setminus \{(u-14)/2\}$ into pairs $\{x_i, y_i\}$, $1 \le i \le u$, such that $\{y_i - x_i : 1 \le i \le u\} = [17, 16+u]$.

- u=38:

(13,41)	(14,43)	(15,45)	(16,47)	(17,58)	(19,61)	(21,60)	(23,63)	(25,62)	(27,65)	(29,64),
(10,35)	(11,59)	(31,75)	(32,77)	(33,76)	(36,68)	(22,73)	(24,44)	(28,50)	(34,53)	(1,37),
(2,49)	(3,56)	(4,38)	(5,54)	(6,39)	(7,57)	(8,26)	(9,55)	(18,70)	(20,74)	(30,51),
(40,67)	(42,66)	(46,72)	(48,71)	(52,69)						

- u=42:

(10,36) (21,55) (23,58) (25,61) (27,64) (29,74) (31,77) (33,80) (35,78) (37,79) (39,83),
(40,81) (43,82) (45,85) (46,84) (11,59) (12,62) (13,65) (15,66) (16,69) (17,50) (19,68),
(38,70) (41,71) (42,73) (1,56) (2,20) (3,57) (4,60) (5,22) (6,63) (7,26) (8,30),
(9,67) (18,47) (24,49) (28,51) (32,53) (34,54) (44,72) (48,75) (52,76)

- u=46:

(21,55) (23,58) (25,61) (27,64) (29,67) (31,70) (33,79) (35,84) (37,85) (39,86) (41,91)
(43,87) (45,88) (47,89) (48,93) (50,90) (51,92) (13,65) (14,68) (15,71) (17,72) (19,76)
(40,69) (42,73) (44,74) (46,78) (49,82) (52,80) (11,62) (1,20) (2,60) (3,63) (4,66)
(5,26) (6,59) (7,24) (8,28) (9,32) (10,34) (12,38) (18,77) (22,83) (30,57) (36,54)
(53,75) (56,81)

- u=50:

(17,50) (22,56) (23,58) (25,61) (27,64) (29,67) (31,70) (33,73) (35,76) (37,79) (39,82)
(41,85) (43,88) (45,93) (47,97) (49,100) (51,98) (52,101) (53,99) (16,74) (40,96) (42,94)
(44,72) (46,75) (48,80) (54,84) (55,86) (11,38) (12,77) (20,83) (24,90) (1,19) (2,21)
(3,26) (4,57) (5,60) (6,68) (7,66) (8,28) (9,63) (10,71) (13,30) (14,78) (15,36)
(32,92) (34,91) (59,81) (62,87) (65,89) (69,95)

- u=54:

(22,56) (23,58) (25,61) (27,64) (29,67) (31,70) (33,73) (35,76) (51,107) (52,109) (53,103)
(54,105) (55,108) (57,84) (60,90) (45,93) (47,99) (48,102) (49,104) (17,85) (30,91) (34,96)
(50,78) (65,98) (68,92) (69,87) (72,97) (77,106) (37,101) (39,81) (41,88) (1,18) (2,21)
(3,24) (4,26) (5,36) (6,66) (7,74) (8,71) (9,32) (10,80) (11,43) (12,38) (13,82)
(14,79) (15,59) (16,75) (19,62) (28,94) (40,86) (42,100) (44,89) (46,95) (63,83)

- u=58:

(37,79) (39,82) (67,102) (69,109) (71,100) (47,94) (49,97) (17,83) (19,87) (44,78) (46,116)
(48,115) (50,80) (53,84) (58,110) (59,117) (60,113) (61,93) (63,96) (26,86) (27,89) (29,90)
(31,95) (33,92) (35,98) (62,99) (64,103) (65,101) (66,104) (51,77) (52,107) (55,106) (56,105)
(1,18) (2,20) (3,23) (4,25) (5,24) (6,28) (7,34) (8,36) (9,74) (10,81) (11,57)
(12,85) (13,70) (14,88) (15,38) (16,72) (21,45) (30,75) (32,76) (40,112) (41,91) (42,111)
(43,68) (54,108) (73,114)

- u=62:

(37,79) (39,82) (67,102) (69,109) (71,100) (47,94) (49,97) (73,114) (74,118) (76,122) (44,78)
(46,116) (48,115) (50,80) (53,84) (58,110) (59,117) (60,113) (61,93) (63,96) (26,86) (27,89)
(29,90) (31,95) (33,92) (35,98) (62,99) (64,103) (65,101) (66,104) (51,77) (15,91) (16,72)
(17,83) (19,87) (1,18) (2,20) (3,22) (4,25) (5,28) (6,30) (7,57) (8,85) (9,34)
(10,81) (11,56) (12,40) (13,68) (14,41) (21,43) (23,88) (32,107) (36,108) (38,111) (42,120)
(45,119) (52,106) (54,123) (55,112) (70,121) (75,124) (105,125)

- u=66:

(37,79) (39,82) (67,102) (69,109) (71,100) (47,94) (49,97) (19,74) (21,89) (57,133) (63,128)
(70,132) (72,129) (81,131) (44,78) (46,116) (48,115) (50,80) (53,84) (58,110) (73,127) (75,119)
(76,121) (77,123) (83,124) (56,117) (59,130) (35,98) (62,99) (64,103) (65,101) (66,104) (60,93)
(61,112) (42,91) (1,18) (2,20) (3,22) (4,24) (5,27) (6,29) (7,28) (8,32) (9,34)
(10,36) (11,38) (12,40) (13,85) (14,88) (15,92) (16,95) (17,86) (23,96) (25,106) (30,108)
(31,87) (33,113) (41,105) (43,125) (45,120) (51,111) (52,118) (54,107) (55,114) (68,126) (90,122)

- u=70:

(85,141) (87,140) (5,26) (67,102) (69,109) (71,100) (47,94) (49,97) (19,74) (21,89) (57,133)
(63,128) (70,132) (72,129) (81,131) (44,78) (46,116) (48,115) (50,80) (53,84) (58,110) (73,127)
(75,119) (76,121) (77,123) (83,124) (56,117) (59,130) (35,98) (62,99) (64,103) (65,101) (66,104)
(60,93) (61,112) (42,91) (32,111) (33,106) (41,107) (43,120) (1,18) (2,20) (3,22) (4,24)
(105,137) (6,29) (7,31) (8,30) (9,34) (10,68) (11,38) (12,40) (13,39) (14,96) (15,90)
(16,88) (17,86) (23,82) (25,108) (27,113) (36,114) (37,122) (45,125) (51,135) (52,126) (54,118)
(55,136) (79,139) (92,134) (95,138)

- u=74:

(85,141) (87,140) (5,26) (67,102) (69,109) (71,100) (88,147) (90,148) (23,95) (57,133) (63,128)
(70,132) (72,129) (81,131) (44,78) (46,116) (48,115) (50,80) (122,145) (21,49) (25,114) (27,82)
(29,113) (37,127) (39,108) (75,119) (76,121) (77,123) (83,124) (56,117) (59,130) (35,98) (62,99)
(64,103) (65,101) (66,104) (60,93) (61,112) (42,91) (32,111) (33,106) (41,107) (43,120) (52,126)
(68,149) (79,143) (84,136) (1,18) (2,20) (3,22) (4,24) (105,137) (6,28) (7,34) (8,51)
(9,89) (10,36) (11,58) (12,54) (13,38) (14,45) (15,97) (16,94) (17,92) (19,73) (31,118)
(40,125) (47,135) (53,139) (55,138) (74,142) (86,146) (96,144) (110,134)

- u=78:

(85,141) (87,140) (33,106) (41,107) (69,109) (71,100) (88,147) (90,148) (50,80) (57,133) (63,128)
(70,132) (53,134) (55,137) (27,110) (44,78) (46,116) (48,115) (122,145) (23,95) (21,49) (25,114)
(54,139) (68,142) (79,157) (29,113) (37,127) (39,108) (11,102) (123,155) (13,105) (83,124) (56,117)
(59,130) (35,98) (62,99) (64,103) (65,101) (66,104) (60,93) (61,112) (42,91) (86,138) (89,153)
(92,152) (96,144) (73,150) (74,149) (77,121) (81,126) (82,129) (118,143) (2,20) (135,156) (4,24)
(5,47) (6,28) (7,75) (8,58) (9,36) (10,34) (14,40) (120,151) (16,51) (17,97) (19,76)
(26,72) (30,84) (31,119) (38,125) (43,136) (45,131) (52,146) (111,154) (1,18) (15,94) (12,67)
(3,22)

- u=82:

(85,141) (87,140) (33,106) (41,107) (69,109) (71,100) (88,147) (90,148) (122,145) (57,133)
(63,128) (70,132) (53,134) (55,137) (27,110) (44,78) (46,116) (48,115) (50,80) (23,95)
(21,49) (25,114) (54,139) (68,142) (79,157) (29,113) (37,127) (39,108) (11,102) (12,67)
(13,105) (83,124) (56,117) (59,130) (35,98) (62,99) (64,103) (65,101) (26,119) (97,165)
(104,151) (112,163) (38,125) (45,131) (47,135) (86,138) (89,153) (92,152) (96,144) (91,136)
(94,143) (118,164) (123,158) (1,18) (2,20) (3,22) (4,24) (5,30) (6,28) (7,31)
(8,40) (9,84) (10,60) (14,111) (15,36) (16,42) (17,74) (19,73) (32,75) (43,76)
(51,146) (52,150) (58,154) (61,155) (66,93) (72,149) (77,156) (81,161) (82,126) (120,162)
(121,159) (129,160)

- u=86:

(85,141) (87,140) (33,106) (41,107) (69,109) (71,100) (88,147) (90,148) (122,145) (57,133)
(63,128) (70,132) (53,134) (55,137) (27,110) (44,78) (46,116) (48,115) (50,80) (127,158)
(150,171) (25,114) (54,139) (68,142) (79,157) (29,113) (39,108) (11,102) (12,67) (13,105)
(83,124) (56,117) (59,130) (35,98) (17,60) (91,129) (92,136) (96,160) (103,149) (118,166)
(120,155) (61,161) (65,101) (26,119) (97,165) (104,151) (112,163) (38,125) (45,131) (47,135)
(86,138) (93,143) (75,172) (76,170) (77,156) (84,164) (1,18) (2,20) (3,22) (4,24)
(5,30) (6,28) (7,31) (8,34) (9,42) (10,37) (14,51) (15,111) (16,58) (19,64)
(21,49) (23,121) (32,81) (40,72) (43,144) (52,154) (62,152) (66,126) (73,168) (74,173)
(82,159) (89,146) (94,169) (95,167) (99,153) (123,162)

- u=90:

(85,141) (112,163) (41,107) (69,109) (71,100) (88,147) (90,148) (98,173) (121,152) (36,93)
(143,180) (73,171) (70,132) (45,131) (47,135) (27,110) (44,78) (46,116) (48,115) (50,80)
(52,153) (62,159) (74,170) (76,175) (25,114) (54,139) (68,142) (79,157) (29,113) (39,108)
(11,102) (12,67) (40,144) (83,124) (56,117) (59,130) (66,168) (72,178) (82,177) (89,179)
(94,181) (23,128) (95,167) (99,126) (123,162) (91,129) (92,136) (96,160) (103,149) (118,166)
(81,158) (86,138) (87,169) (105,150) (125,174) (133,161) (97,165) (104,151) (33,106) (1,18)
(2,20) (3,22) (4,24) (5,26) (6,28) (7,30) (8,32) (9,34) (10,42) (13,55)
(14,57) (15,65) (16,49) (17,77) (19,119) (21,75) (31,84) (35,127) (37,63) (43,122)
(51,154) (53,146) (58,134) (60,140) (61,155) (64,145) (101,164) (111,176) (120,156) (137,172)

- u=94:

(85,141) (17,53) (51,77) (60,163) (65,146) (97,165) (41,107) (69,109) (71,100) (88,147)
(90,148) (98,173) (121,152) (36,93) (143,180) (73,171) (70,132) (45,131) (47,135) (46,116)
(48,115) (50,80) (52,153) (62,159) (74,170) (76,175) (25,114) (54,139) (68,142) (79,157)
(29,113) (39,108) (11,102) (83,124) (56,117) (59,130) (84,188) (104,151) (72,178) (82,177)
(89,179) (94,181) (23,128) (95,167) (99,126) (123,162) (91,129) (92,136) (96,160) (103,149)
(118,166) (81,158) (86,138) (87,169) (61,140) (63,156) (66,168) (75,184) (105,155) (1,18)
(2,20) (3,22) (4,24) (5,26) (6,28) (7,30) (8,32) (9,42) (10,38) (12,55)
(13,64) (14,67) (15,49) (16,58) (19,44) (21,101) (27,119) (31,125) (33,133) (34,144)
(35,111) (37,145) (43,106) (57,164) (78,161) (110,183) (112,172) (120,174) (122,187) (127,176)
(134,189) (137,182) (150,185) (154,186)

- u=98:

(85,141) (111,174) (112,172) (119,192) (105,197) (106,189) (110,186) (120,185) (122,176) (127,182)
(60,163) (65,146) (97,165) (41,107) (69,109) (71,100) (88,147) (90,148) (98,173) (121,152)
(36,93) (143,180) (73,171) (70,132) (45,131) (47,135) (46,116) (48,115) (50,80) (52,153)
(62,159) (74,170) (76,175) (25,114) (54,139) (68,142) (79,157) (29,113) (39,108) (11,102)
(83,124) (56,117) (59,130) (84,188) (104,151) (72,178) (82,177) (89,179) (94,181) (23,128)
(95,167) (99,126) (123,162) (91,129) (92,136) (96,160) (103,149) (118,166) (81,158) (1,18)
(2,20) (3,22) (4,24) (5,26) (6,28) (7,30) (8,32) (9,34) (10,38) (12,44)
(13,49) (14,40) (15,57) (16,51) (17,66) (19,64) (21,55) (27,140) (31,125) (33,133)
(35,87) (37,145) (43,155) (53,164) (58,138) (61,168) (63,156) (67,169) (75,154) (77,191)
(78,187) (86,196) (101,183) (134,184) (137,190) (144,195) (150,193) (161,194)

- u=102:

(85,141) (111,174) (112,172) (119,192) (105,197) (106,189) (110,186) (120,185) (122,176) (127,182)
(60,163) (65,146) (97,165) (41,107) (69,109) (71,100) (88,147) (90,148) (98,173) (121,152)
(36,93) (143,180) (73,171) (70,132) (45,131) (101,195) (103,183) (118,200) (125,204) (52,153)
(62,159) (74,170) (76,175) (25,114) (54,139) (68,142) (79,157) (29,113) (39,108) (11,102)
(83,124) (56,117) (59,130) (84,188) (104,151) (72,178) (82,177) (89,179) (94,181) (23,128)
(95,167) (99,126) (123,162) (91,129) (92,136) (81,158) (27,77) (31,63) (33,140) (35,144)
(37,145) (1,18) (2,20) (3,22) (4,24) (5,26) (6,28) (7,30) (8,32) (9,34)
(10,38) (12,42) (13,46) (14,40) (15,49) (16,51) (17,53) (19,64) (21,67) (43,154)
(47,164) (48,96) (50,138) (55,169) (57,150) (58,160) (61,161) (66,184) (75,187) (78,194)
(80,193) (86,196) (87,202) (115,166) (116,168) (133,203) (134,201) (135,199) (137,190) (149,191)
(155,198) (156,205)

• u=106:

(85,141) (111,174) (112,172) (119,192) (105,197) (106,189) (110,186) (120,185) (122,176) (127,182)
(60,163) (65,146) (97,165) (41,107) (115,203) (133,184) (134,198) (135,205) (98,173) (121,152)
(36,93) (143,180) (73,171) (70,132) (45,131) (101,195) (103,183) (118,200) (125,204) (61,161)
(62,175) (63,160) (155,207) (25,114) (87,208) (90,209) (100,212) (109,211) (39,108) (11,102)
(83,124) (56,117) (59,130) (84,188) (104,151) (72,178) (82,177) (89,179) (94,181) (23,128)
(95,167) (99,126) (123,162) (91,129) (92,136) (81,158) (113,159) (116,201) (137,196) (33,140)
(35,144) (37,145) (1,18) (2,20) (3,22) (4,24) (5,26) (6,28) (7,30) (8,32)
(9,34) (10,38) (12,42) (13,47) (14,40) (15,44) (16,48) (17,50) (19,54) (21,57)
(27,67) (29,71) (31,79) (43,88) (49,142) (51,169) (52,168) (53,154) (55,166) (58,157)
(64,138) (66,150) (68,164) (69,191) (74,194) (75,190) (76,193) (77,187) (78,156) (80,147)
(86,139) (96,210) (148,206) (149,199) (153,202) (170,213)

• u=110:

(124,219) (111,174) (112,172) (119,192) (105,197) (106,189) (110,186) (120,185) (122,176) (127,182)
(60,163) (65,146) (97,165) (41,107) (115,203) (133,184) (134,198) (135,205) (98,173) (121,152)
(82,187) (89,212) (102,214) (45,131) (101,195) (103,183) (118,200) (125,204) (61,161) (62,175)
(63,160) (155,207) (87,208) (90,209) (83,199) (86,170) (88,210) (96,216) (117,166) (142,220)
(149,202) (84,188) (104,151) (139,213) (150,217) (156,206) (94,181) (95,167) (99,126) (123,162)
(91,129) (92,136) (81,158) (113,159) (116,201) (137,196) (33,140) (35,144) (77,191) (93,211)
(108,218) (114,171) (59,130) (66,164) (68,194) (80,141) (1,18) (2,20) (3,22) (4,24)
(5,26) (6,28) (7,30) (8,32) (9,34) (10,36) (11,39) (12,42) (13,46) (14,43)
(15,47) (16,50) (17,52) (19,55) (21,58) (23,64) (25,73) (27,67) (29,154) (31,74)
(37,79) (38,153) (40,85) (44,145) (49,148) (51,147) (53,109) (54,143) (56,180) (57,168)
(69,138) (70,132) (71,179) (72,178) (75,177) (76,193) (78,169) (100,190) (128,221) (157,215)

• u=114:

(124,219) (111,174) (112,172) (119,192) (105,197) (106,189) (110,186) (120,185) (122,176) (47,128)
(49,107) (52,148) (54,178) (92,208) (129,222) (154,223) (115,203) (133,184) (134,198) (135,205)
(98,173) (121,152) (82,187) (89,212) (102,214) (85,147) (86,170) (97,145) (101,195) (103,183)
(118,200) (125,204) (61,161) (114,171) (127,182) (63,160) (155,207) (79,190) (100,227) (109,226)
(132,221) (88,210) (96,216) (117,166) (142,220) (149,202) (84,188) (104,151) (139,213) (150,217)
(156,206) (94,181) (95,167) (93,211) (108,218) (71,177) (72,180) (73,141) (90,209) (138,228)
(168,224) (113,159) (116,201) (137,196) (33,140) (35,144) (1,18) (2,20) (3,22) (4,24)
(5,26) (6,28) (7,30) (8,32) (9,34) (10,36) (11,38) (12,40) (13,42) (14,44)
(15,48) (16,51) (17,53) (19,56) (21,55) (23,62) (25,57) (27,65) (29,69) (31,75)
(37,136) (39,165) (41,162) (43,157) (45,158) (46,91) (58,99) (59,130) (60,163) (64,194)
(66,143) (67,153) (68,193) (70,199) (74,175) (76,191) (77,179) (78,169) (80,146) (81,123)
(83,126) (87,215) (131,229) (164,225)

• u=118:

(124,219) (111,174) (112,172) (119,192) (105,197) (106,189) (110,186) (120,185) (122,176) (47,128)
(54,178) (92,208) (129,222) (154,223) (115,203) (133,184) (134,198) (135,205) (98,173) (121,152)
(82,187) (89,212) (102,214) (85,147) (86,170) (97,145) (101,195) (103,183) (118,200) (125,204)
(83,215) (130,231) (141,237) (127,182) (63,160) (155,207) (79,190) (100,227) (109,226) (132,221)
(88,210) (96,216) (117,166) (142,220) (149,202) (84,188) (104,151) (139,213) (150,217) (156,206)
(94,181) (77,162) (80,211) (93,161) (95,167) (116,175) (123,230) (108,218) (107,232) (126,229)
(131,233) (143,234) (87,164) (91,224) (114,235) (146,236) (169,225) (1,18) (2,20) (3,22)
(4,24) (5,26) (6,28) (7,30) (8,32) (9,34) (10,36) (11,38) (12,40) (13,42)
(14,44) (15,48) (16,50) (17,49) (19,55) (21,56) (23,60) (25,64) (27,65) (29,69)
(31,72) (33,75) (35,78) (37,81) (39,157) (41,99) (43,171) (45,90) (46,159) (51,165)
(53,153) (57,163) (58,144) (59,158) (61,180) (62,191) (66,196) (67,201) (68,177) (70,168)
(71,179) (73,199) (74,140) (76,137) (113,228) (136,193) (138,209) (148,194)

• u=122:

(124,219) (111,174) (112,172) (119,192) (105,197) (106,189) (110,186) (120,185) (122,176) (129,222)
(154,223) (115,203) (133,184) (134,198) (135,205) (113,242) (128,241) (138,244) (144,243) (89,212)
(102,214) (101,225) (103,238) (163,209) (165,245) (153,228) (159,240) (168,239) (61,199) (64,148)
(66,196) (118,200) (125,204) (83,215) (130,231) (141,237) (127,182) (63,160) (155,207) (79,190)
(100,227) (109,226) (132,221) (88,210) (96,216) (117,166) (142,220) (149,202) (84,188) (104,151)
(139,213) (150,217) (156,206) (94,181) (77,162) (80,211) (93,161) (95,167) (116,175) (123,230)
(108,218) (107,232) (126,229) (131,233) (143,234) (87,164) (91,224) (121,236) (136,193) (140,201)
(1,18) (2,20) (3,22) (4,24) (5,26) (6,28) (7,30) (8,32) (9,34) (10,36)
(11,38) (12,40) (13,42) (14,44) (15,46) (16,48) (17,50) (19,53) (21,56) (23,59)
(25,62) (27,65) (29,68) (31,71) (33,74) (35,78) (37,81) (39,97) (41,146) (43,171)
(45,90) (47,173) (49,158) (51,99) (52,170) (55,145) (57,157) (58,114) (60,194) (67,183)
(69,177) (70,191) (72,208) (73,187) (75,169) (76,195) (82,180) (85,147) (86,152) (92,178)
(98,235) (137,179)

- u=126:

(124,219)	(111,174)	(112,172)	(119,192)	(105,197)	(89,169)	(90,180)	(92,178)	(120,185)	(122,176)
(129,222)	(154,223)	(115,203)	(133,184)	(134,198)	(135,205)	(113,242)	(128,241)	(138,244)	(144,243)
(91,212)	(97,236)	(110,194)	(140,249)	(102,214)	(101,225)	(103,238)	(136,193)	(152,246)	(153,228)
(159,240)	(168,239)	(121,247)	(137,252)	(145,250)	(146,208)	(195,251)	(125,204)	(83,215)	(130,231)
(141,237)	(127,182)	(63,160)	(155,207)	(79,190)	(100,227)	(109,226)	(87,201)	(98,235)	(163,245)
(114,248)	(187,253)	(117,166)	(142,220)	(149,202)	(84,188)	(104,151)	(139,213)	(150,217)	(156,206)
(94,181)	(77,162)	(80,211)	(93,161)	(95,167)	(116,175)	(123,230)	(108,218)	(107,232)	(126,229)
(131,233)	(1,18)	(2,20)	(3,22)	(4,24)	(5,26)	(6,28)	(7,30)	(8,32)	(9,34)
(10,36)	(11,38)	(12,40)	(13,42)	(14,44)	(15,46)	(16,48)	(17,50)	(19,53)	(21,57)
(23,58)	(25,62)	(27,65)	(29,68)	(31,71)	(33,74)	(35,78)	(37,81)	(39,85)	(41,86)
(43,132)	(45,106)	(47,170)	(49,177)	(51,173)	(52,171)	(54,96)	(55,191)	(59,200)	(60,143)
(61,179)	(64,164)	(66,186)	(67,165)	(69,199)	(70,147)	(72,210)	(73,189)	(75,183)	(76,216)
(82,224)	(88,221)	(99,157)	(118,209)	(148,196)	(158,234)				

- u=130:

(124,219)	(111,174)	(112,172)	(119,192)	(105,197)	(127,173)	(155,256)	(183,231)	(120,185)	(122,176)
(129,222)	(154,223)	(115,203)	(133,184)	(134,198)	(135,205)	(113,242)	(128,241)	(138,244)	(144,243)
(91,212)	(97,236)	(110,194)	(140,249)	(102,214)	(101,225)	(103,238)	(136,193)	(152,246)	(153,228)
(159,240)	(168,239)	(121,247)	(137,252)	(145,250)	(146,208)	(195,251)	(125,204)	(118,254)	(148,224)
(157,257)	(164,260)	(132,255)	(143,261)	(160,237)	(100,227)	(109,226)	(87,201)	(98,235)	(163,245)
(114,248)	(187,253)	(117,166)	(142,220)	(149,202)	(84,188)	(104,151)	(139,213)	(150,217)	(156,206)
(81,189)	(82,171)	(85,182)	(88,199)	(89,169)	(90,234)	(92,178)	(96,179)	(116,175)	(123,230)
(108,218)	(1,18)	(2,20)	(3,22)	(4,24)	(5,26)	(6,28)	(7,30)	(8,32)	(9,34)
(10,36)	(11,38)	(12,40)	(13,42)	(14,44)	(15,46)	(16,48)	(17,50)	(19,53)	(21,56)
(23,59)	(25,62)	(27,65)	(29,68)	(31,71)	(33,74)	(35,77)	(37,80)	(39,83)	(41,86)
(43,95)	(45,106)	(47,177)	(49,165)	(51,170)	(52,180)	(54,200)	(55,158)	(57,147)	(60,162)
(61,186)	(63,161)	(64,196)	(66,207)	(67,209)	(69,141)	(70,190)	(72,215)	(73,131)	(75,130)
(76,221)	(78,216)	(79,210)	(93,233)	(94,181)	(99,232)	(107,229)	(126,211)	(167,258)	(191,259)

E Partition the set $[1, 2u+1] \setminus \{(u-30)/2\}$ into pairs

Here for each $u \in [68, 259]$ and $u \equiv 2 \pmod 4$, we give a partition of the set $[1, 2u+1] \setminus \{(u-30)/2\}$ into pairs $\{x_i, y_i\}$, $1 \le i \le u$, such that $\{y_i - x_i : 1 \le i \le u\} = [33, 32+u]$.

- u=70:

(38,81)	(44,131)	(45,135)	(50,96)	(52,92)	(54,116)	(32,95)	(33,114)	(35,117)	(47,130)
(55,132)	(57,136)	(51,124)	(18,68)	(19,70)	(63,98)	(65,120)	(13,58)	(56,110)	(53,94)
(15,99)	(16,101)	(17,105)	(48,137)	(14,106)	(21,87)	(22,83)	(26,91)	(27,118)	(30,97)
(31,90)	(37,107)	(41,109)	(43,80)	(46,85)	(60,113)	(77,115)	(59,134)	(61,139)	(28,86)
(64,133)	(72,119)	(73,121)	(74,123)	(67,111)	(69,126)	(1,34)	(2,62)	(3,89)	(4,84)
(5,100)	(6,102)	(7,108)	(8,79)	(9,103)	(10,82)	(11,104)	(12,112)	(23,75)	(24,88)
(25,122)	(29,128)	(36,78)	(39,141)	(40,138)	(42,76)	(49,125)	(66,140)	(71,127)	(93,129)

- u=74:

(39,104)	(51,117)	(38,143)	(54,130)	(56,113)	(60,108)	(63,127)	(65,121)	(31,125)	(33,136)
(35,135)	(55,141)	(13,90)	(14,89)	(19,116)	(72,131)	(77,139)	(83,144)	(84,138)	(91,134)
(25,62)	(44,112)	(45,133)	(46,106)	(53,122)	(58,105)	(59,103)	(67,107)	(68,109)	(17,80)
(21,93)	(26,110)	(61,97)	(64,145)	(69,118)	(28,123)	(32,71)	(48,137)	(36,129)	(42,100)
(73,147)	(76,149)	(78,148)	(81,119)	(41,128)	(43,114)	(10,111)	(11,66)	(40,86)	(47,146)
(1,34)	(2,87)	(3,99)	(4,102)	(5,96)	(6,88)	(7,52)	(8,98)	(9,115)	(12,92)
(15,49)	(16,95)	(18,70)	(20,124)	(23,101)	(24,74)	(27,94)	(29,82)	(30,132)	(37,79)
(50,142)	(57,140)	(75,126)	(85,120)						

- u=78:

(39,104)	(51,117)	(38,143)	(54,130)	(56,113)	(60,108)	(63,127)	(65,121)	(31,125)	(33,136)
(35,135)	(55,141)	(61,97)	(64,145)	(19,116)	(72,131)	(77,139)	(83,144)	(84,138)	(91,134)
(57,96)	(85,152)	(45,133)	(46,106)	(53,122)	(58,105)	(59,103)	(67,107)	(68,109)	(17,80)
(21,93)	(26,110)	(40,74)	(47,155)	(69,118)	(28,123)	(49,156)	(66,157)	(36,129)	(42,100)
(73,147)	(76,149)	(78,148)	(81,119)	(41,128)	(43,114)	(10,111)	(52,154)	(70,153)	(16,101)
(27,137)	(11,115)	(12,102)	(30,126)	(1,34)	(2,37)	(3,48)	(4,79)	(5,82)	(6,86)
(7,89)	(8,50)	(9,98)	(13,112)	(14,92)	(15,94)	(18,71)	(20,88)	(22,120)	(23,132)
(25,62)	(29,75)	(32,124)	(44,150)	(87,142)	(90,140)	(95,146)	(99,151)		

• u=82:

(39,104) (51,117) (38,143) (54,130) (56,113) (60,108) (63,127) (65,121) (31,125) (33,136)
(35,135) (55,141) (17,101) (18,124) (64,145) (19,116) (72,131) (77,139) (83,144) (84,138)
(91,134) (29,142) (73,158) (85,152) (45,133) (46,106) (53,122) (58,105) (59,103) (67,107)
(68,109) (93,165) (22,120) (15,126) (16,115) (32,146) (47,155) (69,118) (28,123) (49,156)
(66,157) (36,129) (42,100) (95,150) (75,164) (79,159) (44,148) (81,119) (41,128) (43,114)
(86,161) (88,151) (90,160) (94,147) (23,132) (70,153) (1,34) (2,37) (3,40) (4,96)
(5,82) (6,48) (7,89) (8,87) (9,111) (10,78) (11,57) (12,102) (13,52) (14,110)
(20,71) (21,99) (24,74) (25,98) (27,61) (30,140) (50,162) (62,163) (76,112) (80,154)
(92,137) (97,149)

• u=86:

(39,104) (51,117) (38,143) (54,130) (56,113) (60,108) (63,127) (65,121) (31,125) (33,136)
(35,135) (61,171) (82,172) (18,124) (64,145) (19,116) (72,131) (77,139) (83,144) (84,138)
(91,134) (29,142) (73,158) (85,152) (45,133) (46,106) (53,122) (58,105) (59,103) (67,107)
(68,109) (93,165) (22,120) (71,173) (74,169) (32,146) (47,155) (69,118) (49,156) (66,157)
(36,129) (42,100) (95,150) (75,164) (79,159) (44,148) (81,119) (41,128) (43,114) (86,161)
(88,151) (90,160) (78,170) (89,166) (70,153) (94,167) (102,154) (27,80) (30,147) (1,34)
(2,37) (3,40) (4,50) (5,101) (6,57) (7,52) (8,76) (9,110) (10,92) (11,97)
(12,111) (13,87) (14,123) (15,99) (16,55) (17,96) (20,98) (21,132) (23,141) (24,140)
(25,137) (26,62) (48,163) (112,162) (115,149) (126,168)

• u=90:

(39,104) (51,117) (38,143) (54,130) (56,113) (60,108) (63,127) (65,121) (31,125) (33,136)
(35,135) (61,171) (82,172) (18,124) (64,145) (16,115) (17,132) (77,139) (83,144) (84,138)
(91,134) (29,142) (73,158) (85,152) (45,133) (46,106) (53,122) (58,105) (59,103) (67,107)
(68,109) (93,165) (22,120) (71,173) (74,169) (32,146) (47,155) (81,177) (23,101) (66,157)
(36,129) (42,100) (95,150) (75,164) (79,159) (69,176) (96,175) (41,128) (43,114) (86,161)
(88,151) (90,160) (78,170) (89,166) (70,153) (94,167) (26,148) (76,180) (1,34) (2,37)
(3,40) (4,49) (5,44) (6,48) (7,116) (8,57) (9,62) (10,111) (11,123) (12,98)
(13,131) (14,50) (15,112) (19,87) (20,137) (21,72) (24,140) (25,99) (27,147) (28,149)
(52,163) (55,174) (80,162) (92,126) (97,181) (102,154) (110,156) (118,168) (119,178) (141,179)

• u=94:

(39,104) (51,117) (38,143) (54,130) (56,113) (60,108) (63,127) (65,121) (31,125) (33,136)
(35,135) (61,171) (82,172) (18,124) (64,145) (102,186) (114,167) (77,139) (83,144) (99,178)
(30,154) (36,129) (29,142) (73,158) (85,152) (45,133) (46,106) (53,122) (58,105) (59,103)
(67,107) (68,109) (93,165) (110,156) (71,173) (74,169) (47,155) (81,177) (116,168) (87,184)
(91,182) (42,100) (95,150) (80,181) (92,163) (79,159) (69,176) (111,162) (112,146) (96,183)
(98,157) (86,161) (88,151) (90,160) (78,170) (89,166) (70,153) (101,179) (76,185) (119,187)
(27,147) (1,34) (2,37) (3,40) (4,43) (5,41) (6,44) (7,49) (8,57) (9,52)
(10,84) (11,115) (12,123) (13,131) (14,126) (15,97) (16,132) (17,62) (19,138) (20,118)
(21,120) (22,72) (23,137) (24,141) (25,140) (26,148) (28,149) (48,174) (50,175) (55,128)
(66,189) (75,164) (94,180) (134,188)

• u=98:

(39,104) (51,117) (38,143) (54,130) (56,113) (60,108) (63,127) (65,121) (31,125) (33,136)
(35,135) (61,171) (82,172) (97,195) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178)
(30,154) (132,190) (148,197) (73,158) (85,152) (45,133) (46,106) (53,122) (58,105) (78,194)
(89,166) (68,109) (93,165) (66,192) (67,185) (111,189) (18,138) (116,168) (32,160) (47,155)
(81,177) (23,115) (87,184) (94,193) (19,134) (107,188) (80,181) (92,163) (79,159) (69,176)
(15,124) (112,146) (96,183) (98,157) (86,161) (88,151) (71,173) (72,191) (70,153) (50,180)
(120,174) (123,156) (75,170) (76,187) (91,182) (1,36) (2,40) (3,42) (4,41) (5,48)
(6,52) (7,43) (8,59) (9,49) (10,55) (11,128) (12,141) (13,140) (14,64) (16,84)
(17,131) (20,142) (21,103) (22,126) (24,147) (25,95) (26,119) (27,100) (28,149) (29,118)
(37,162) (44,150) (57,169) (62,175) (74,129) (90,164) (101,145) (137,179)

• u=102:

(39,104) (51,117) (38,143) (54,130) (56,113) (60,108) (63,127) (65,121) (31,125) (33,136)
(35,135) (61,171) (82,172) (97,195) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178)
(30,154) (132,190) (148,197) (95,150) (115,183) (45,133) (46,106) (53,122) (58,105) (78,194)
(89,166) (68,109) (93,165) (66,192) (67,185) (111,189) (34,168) (62,147) (32,160) (47,155)
(100,204) (119,201) (87,184) (94,193) (19,134) (107,188) (81,177) (84,205) (79,159) (80,181)
(85,202) (101,175) (126,199) (27,158) (28,141) (88,151) (71,173) (72,191) (70,153) (50,180)
(120,174) (123,156) (75,170) (76,187) (91,182) (98,157) (112,164) (96,163) (86,179) (116,161)
(1,37) (2,40) (3,42) (4,41) (5,48) (6,118) (7,49) (8,52) (9,44) (10,142)
(11,131) (12,137) (13,59) (14,64) (15,55) (16,103) (17,92) (18,124) (20,90) (21,128)
(22,145) (23,74) (24,146) (25,152) (26,140) (29,138) (43,176) (57,149) (69,198) (73,162)
(129,200) (169,203)

- u=106:

(39,104) (51,117) (54,130) (56,113) (60,108) (63,127) (65,121) (31,125) (33,136) (35,135)
(61,171) (82,172) (97,195) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178) (30,154)
(132,190) (148,197) (95,150) (115,183) (116,209) (23,152) (24,131) (58,105) (78,194) (89,166)
(68,109) (93,165) (66,192) (67,185) (111,189) (34,168) (62,147) (32,160) (47,155) (100,204)
(119,201) (87,184) (94,193) (19,134) (107,188) (81,177) (84,205) (79,159) (80,181) (85,202)
(101,175) (126,199) (27,158) (28,141) (88,151) (71,173) (72,191) (70,153) (50,180) (120,174)
(123,156) (96,210) (103,208) (128,203) (91,182) (98,157) (112,164) (129,179) (142,213) (133,200)
(1,36) (2,40) (3,37) (4,41) (5,44) (6,42) (7,49) (8,48) (9,52) (10,55)
(11,57) (12,124) (13,64) (14,146) (15,140) (16,122) (17,86) (18,106) (20,90) (21,143)
(22,149) (25,145) (26,161) (29,73) (43,176) (45,137) (46,169) (53,162) (59,170) (69,206)
(74,212) (75,211) (76,163) (92,187) (118,207) (138,198)

- u=110:

(39,104) (51,117) (54,130) (56,113) (60,108) (63,127) (65,121) (31,125) (33,136) (35,135)
(27,168) (28,162) (29,160) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178) (44,176)
(132,190) (48,155) (49,140) (74,216) (115,183) (116,209) (148,207) (122,211) (58,105) (78,194)
(89,166) (68,109) (93,165) (66,192) (34,147) (42,150) (133,200) (95,220) (98,221) (145,187)
(69,161) (100,204) (119,201) (87,184) (94,193) (19,134) (107,188) (81,177) (84,205) (79,159)
(80,181) (85,202) (101,175) (126,199) (142,197) (92,219) (88,151) (71,173) (72,191) (70,153)
(50,180) (120,174) (123,156) (59,154) (118,206) (143,214) (103,208) (128,203) (131,218) (146,215)
(112,164) (129,179) (1,36) (2,38) (3,37) (4,41) (5,43) (6,45) (7,47) (8,52)
(9,55) (10,53) (11,96) (12,90) (13,137) (14,124) (15,64) (16,106) (17,62) (18,138)
(20,149) (21,91) (22,157) (23,141) (24,75) (25,158) (26,163) (30,169) (32,170) (46,182)
(57,171) (61,172) (67,189) (73,213) (76,185) (82,210) (86,198) (97,195) (111,217) (152,212)

- u=114:

(25,149) (26,163) (27,168) (56,113) (60,108) (63,127) (65,121) (132,223) (138,228) (97,226)
(111,217) (28,162) (29,160) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178) (44,176)
(117,195) (124,189) (129,229) (36,169) (38,158) (47,182) (148,207) (122,211) (58,105) (78,194)
(89,166) (68,109) (93,165) (66,192) (34,147) (133,200) (95,220) (98,221) (145,187) (69,161)
(100,204) (119,201) (87,184) (94,193) (19,134) (107,188) (81,177) (84,205) (79,159) (80,181)
(85,202) (101,175) (126,199) (142,197) (92,219) (88,151) (71,173) (106,216) (115,227) (91,210)
(96,224) (120,174) (123,156) (73,209) (104,154) (116,225) (137,222) (143,214) (103,208) (128,203)
(131,218) (146,215) (1,35) (2,37) (3,39) (4,41) (5,43) (6,45) (7,50) (8,48)
(9,53) (10,55) (11,57) (12,61) (13,64) (14,72) (15,67) (16,130) (17,112) (18,140)
(20,86) (21,164) (22,90) (23,141) (24,118) (30,172) (31,171) (32,125) (33,179) (40,170)
(46,191) (49,157) (51,190) (52,155) (54,198) (59,135) (62,150) (70,153) (74,185) (75,213)
(76,183) (82,180) (136,206) (152,212)

- u=118:

(25,149) (26,163) (27,168) (56,113) (60,108) (63,127) (65,121) (132,223) (138,228) (97,226)
(111,217) (28,162) (29,160) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178) (117,195)
(124,189) (129,229) (36,169) (38,158) (47,182) (148,207) (122,211) (58,105) (78,194) (89,166)
(68,109) (93,165) (66,192) (34,147) (133,200) (95,220) (98,221) (145,187) (69,161) (100,204)
(119,201) (87,184) (94,193) (19,134) (107,188) (81,177) (84,205) (79,159) (80,181) (85,202)
(101,175) (126,199) (142,197) (92,219) (88,151) (71,173) (106,216) (115,227) (91,210) (96,224)
(75,213) (76,212) (123,156) (171,237) (104,154) (116,225) (137,222) (143,214) (103,208) (128,203)
(131,218) (125,232) (136,234) (140,233) (118,206) (141,236) (1,35) (2,37) (3,39) (4,41)
(5,43) (6,45) (7,50) (8,48) (9,53) (10,55) (11,57) (12,61) (13,64) (14,74)
(15,73) (16,86) (17,120) (18,72) (20,152) (21,135) (22,90) (23,153) (24,172) (30,174)
(31,176) (32,179) (33,155) (40,180) (42,185) (46,157) (49,191) (51,190) (52,198) (54,130)
(59,209) (62,170) (67,150) (70,164) (82,231) (112,230) (146,215) (183,235)

- u=122:

(25,149) (26,163) (27,168) (56,113) (60,108) (63,127) (65,121) (132,223) (138,228) (91,174)
(157,245) (111,217) (28,162) (29,160) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178)
(117,195) (124,189) (129,229) (36,169) (38,158) (47,182) (148,207) (122,211) (58,105) (78,194)
(89,166) (68,109) (93,165) (66,192) (34,147) (133,200) (95,220) (98,221) (54,206) (59,210)
(49,177) (119,201) (87,184) (94,193) (19,134) (97,242) (74,170) (88,190) (84,205) (79,159)
(80,181) (85,202) (101,175) (126,199) (142,197) (92,219) (107,188) (135,239) (106,216) (115,227)
(75,224) (76,230) (112,234) (118,236) (141,209) (130,241) (155,231) (123,156) (171,237) (104,154)
(116,225) (137,222) (143,214) (103,208) (128,203) (131,218) (125,232) (183,243) (96,226) (1,35)
(2,37) (3,39) (4,41) (5,43) (6,45) (7,50) (8,48) (9,51) (10,55) (11,57)
(12,61) (13,64) (14,72) (15,67) (16,70) (17,153) (18,81) (20,152) (21,150) (22,120)
(23,176) (24,172) (30,100) (31,173) (32,151) (33,180) (40,179) (42,86) (44,187) (52,198)
(53,191) (62,212) (69,161) (71,215) (73,213) (82,185) (90,204) (136,244) (140,235) (145,238)
(146,240) (164,233)

- u=126:

(46,204) (60,217) (185,248) (108,251) (137,222) (63,127) (65,121) (132,223) (138,228) (91,174)
(157,245) (143,212) (44,183) (29,160) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178)
(117,195) (124,189) (129,229) (81,235) (86,224) (113,243) (148,207) (122,211) (58,105) (78,194)

(89,166) (68,109) (93,165) (66,192) (34,147) (133,200) (136,244) (149,252) (95,220) (98,221)
(54,206) (59,210) (49,177) (119,201) (87,184) (94,193) (19,134) (97,242) (74,170) (88,190)
(84,205) (79,159) (80,181) (85,202) (101,175) (126,199) (142,197) (92,219) (31,155) (32,164)
(33,179) (115,227) (146,238) (112,234) (100,250) (120,213) (118,236) (141,209) (130,241) (73,187)
(123,156) (171,237) (104,154) (116,225) (111,253) (151,249) (103,208) (128,203) (131,218) (51,198)
(53,173) (1,35) (2,37) (3,39) (4,41) (5,43) (6,45) (7,47) (8,50) (9,52)
(10,55) (11,57) (12,56) (13,61) (14,71) (15,64) (16,67) (17,69) (18,72) (20,90)
(21,125) (22,82) (23,152) (24,158) (25,180) (26,145) (27,162) (28,161) (30,140) (36,172)
(38,182) (40,188) (42,191) (62,215) (70,176) (75,216) (76,232) (96,233) (106,246) (107,214)
(135,230) (150,231) (153,247) (163,239) (168,226) (169,240)

- u=130:

(49,198) (60,217) (185,248) (108,251) (137,222) (63,127) (65,121) (132,223) (138,228) (91,174)
(157,245) (143,212) (44,183) (29,160) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178)
(117,195) (124,189) (129,229) (81,235) (86,224) (113,243) (148,207) (43,204) (48,201) (52,192)
(89,166) (68,109) (93,165) (145,259) (150,256) (156,226) (136,244) (149,252) (95,220) (98,221)
(54,206) (59,210) (161,255) (163,258) (87,184) (94,193) (123,257) (97,242) (74,170) (88,190)
(84,205) (79,159) (80,181) (85,202) (101,175) (126,199) (142,197) (92,219) (168,239) (173,230)
(33,179) (115,227) (146,238) (112,234) (100,250) (120,213) (118,236) (141,209) (130,241) (73,233)
(131,218) (140,260) (171,237) (104,154) (116,225) (111,253) (151,249) (103,208) (128,203) (119,200)
(125,261) (122,211) (1,34) (2,36) (3,38) (4,40) (5,42) (6,45) (7,47) (8,46)
(9,51) (10,53) (11,55) (12,57) (13,61) (14,66) (15,62) (16,67) (17,71) (18,64)
(19,134) (20,69) (21,147) (22,82) (23,155) (24,153) (25,135) (26,133) (27,182) (28,187)
(30,177) (31,164) (32,90) (35,191) (37,172) (39,176) (41,169) (56,214) (58,162) (70,232)
(72,216) (75,188) (76,152) (78,194) (96,215) (105,246) (106,254) (107,231) (158,240) (180,247)

- u=134:

(49,198) (60,217) (185,248) (108,251) (137,222) (63,127) (65,121) (132,223) (138,228) (91,174)
(157,245) (143,212) (44,183) (29,160) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178)
(117,195) (124,189) (129,229) (81,235) (86,224) (113,243) (148,207) (43,204) (48,201) (89,166)
(68,109) (93,165) (145,259) (150,256) (156,226) (136,244) (149,252) (95,220) (98,221) (54,206)
(59,210) (161,255) (163,258) (87,184) (94,193) (123,257) (97,242) (74,170) (88,190) (84,205)
(79,159) (80,181) (85,202) (101,175) (126,199) (142,197) (92,219) (168,239) (173,230) (33,179)
(115,227) (146,238) (112,234) (100,250) (120,213) (118,236) (141,209) (130,241) (73,233) (153,269)
(158,247) (171,237) (104,154) (116,225) (111,253) (151,249) (103,208) (128,203) (119,200) (125,261)
(105,268) (107,265) (1,34) (2,36) (3,38) (4,40) (5,42) (6,45) (7,47) (8,46)
(9,51) (10,53) (11,55) (12,57) (13,61) (14,66) (15,62) (16,67) (17,71) (18,64)
(19,106) (20,69) (21,131) (22,82) (23,90) (24,164) (25,191) (26,133) (27,140) (28,147)
(30,177) (31,172) (32,194) (35,155) (37,192) (39,176) (41,169) (50,214) (56,215) (58,182)
(70,218) (72,216) (75,240) (76,232) (78,211) (96,231) (122,254) (134,260) (135,264) (152,267)
(162,266) (180,262) (187,263) (188,246)

- u=138:

(35,176) (43,152) (185,248) (108,251) (137,222) (63,127) (65,121) (132,223) (138,228) (91,174)
(157,245) (143,212) (44,183) (29,160) (110,196) (102,186) (114,167) (77,139) (83,144) (99,178)
(117,195) (124,189) (129,229) (81,235) (86,224) (113,243) (148,207) (177,275) (187,263) (89,166)
(68,109) (93,165) (145,259) (150,256) (156,226) (136,244) (149,252) (95,220) (98,221) (59,210)
(161,255) (163,258) (87,184) (94,193) (123,257) (76,216) (104,214) (88,190) (84,205) (79,159)
(80,181) (56,225) (58,191) (126,199) (142,197) (92,219) (168,239) (173,230) (74,240) (85,246)
(97,249) (146,238) (112,234) (100,250) (120,213) (118,236) (141,209) (130,241) (41,211) (96,260)
(115,227) (131,266) (37,154) (70,232) (158,274) (73,233) (103,208) (128,203) (119,200) (125,261)
(105,268) (107,265) (135,254) (164,277) (155,270) (101,247) (122,218) (111,267) (116,271) (49,206)
(20,71) (21,75) (153,242) (1,34) (2,36) (3,38) (4,40) (5,42) (6,45) (7,47)
(8,46) (9,51) (10,53) (11,55) (12,57) (13,60) (14,62) (15,61) (16,66) (17,69)
(18,67) (19,106) (22,82) (23,151) (24,90) (25,169) (26,171) (27,192) (28,175) (30,134)
(31,198) (32,180) (33,201) (39,188) (48,172) (50,179) (52,194) (64,217) (72,231) (78,215)
(133,253) (140,272) (147,273) (162,269) (170,237) (182,264) (202,276) (204,262)

- u=142:

(28,172) (30,192) (31,198) (185,248) (108,251) (137,222) (63,127) (65,121) (73,242) (78,231)
(91,174) (157,245) (143,212) (44,183) (29,160) (110,196) (102,186) (114,167) (77,139) (83,144)
(99,178) (117,195) (124,189) (129,229) (81,235) (86,224) (113,243) (148,207) (177,275) (187,263)
(89,166) (68,109) (93,165) (145,259) (150,256) (156,226) (136,244) (149,252) (95,220) (98,221)
(59,210) (161,255) (163,258) (87,184) (94,193) (123,257) (76,216) (104,214) (88,190) (84,205)
(79,159) (80,181) (58,191) (180,269) (194,281) (92,219) (168,239) (173,230) (74,240) (85,246)
(97,249) (146,238) (112,234) (100,250) (120,213) (118,236) (141,209) (130,241) (103,262) (132,280)
(115,227) (131,266) (153,285) (171,278) (179,253) (158,274) (147,284) (175,279) (128,203) (119,200)
(125,261) (105,268) (107,265) (135,254) (164,277) (199,272) (215,273) (122,218) (111,267) (96,260)
(101,247) (133,282) (134,276) (48,208) (1,34) (2,36) (3,38) (4,40) (5,42) (6,45)
(7,47) (8,46) (9,51) (10,53) (11,55) (12,57) (13,60) (14,62) (15,61) (16,66)
(17,69) (18,67) (19,70) (20,75) (21,126) (22,82) (23,152) (24,90) (25,182) (26,154)
(27,197) (32,206) (33,142) (35,176) (37,202) (39,211) (41,188) (43,169) (49,204) (50,223)
(52,106) (54,225) (64,232) (71,162) (72,217) (116,233) (138,228) (140,264) (151,271) (155,270)
(170,237) (201,283)

- u=146:

(155,237) (185,248) (108,251) (64,234) (71,215) (65,121) (73,242) (78,231) (91,174) (157,245)
(143,212) (29,160) (87,260) (112,238) (114,167) (94,184) (126,283) (142,289) (117,195) (124,189)
(129,229) (81,235) (86,224) (113,243) (151,246) (177,275) (187,263) (89,166) (68,109) (93,165)
(193,282) (204,271) (156,226) (136,244) (149,252) (95,220) (98,221) (59,210) (139,197) (152,225)
(96,264) (138,287) (123,257) (76,216) (104,214) (88,190) (84,205) (79,159) (80,181) (194,281)
(92,219) (168,239) (173,230) (74,240) (83,180) (99,276) (49,223) (144,286) (100,250) (120,213)
(118,236) (141,209) (130,241) (103,262) (56,201) (115,227) (131,266) (153,285) (171,278) (179,253)
(158,274) (147,284) (175,279) (128,203) (119,200) (125,261) (105,268) (107,265) (135,254) (145,259)
(146,270) (150,256) (134,249) (111,267) (182,291) (101,247) (97,273) (116,178) (132,280) (133,288)
(39,211) (102,277) (110,232) (1,34) (2,36) (3,38) (4,40) (5,42) (6,44) (7,46)
(8,48) (9,51) (10,53) (11,55) (12,57) (13,60) (14,62) (15,61) (16,66) (17,69)
(18,67) (19,70) (20,75) (21,82) (22,106) (23,77) (24,90) (25,85) (26,122) (27,140)
(28,169) (30,192) (31,183) (32,196) (33,172) (35,202) (37,208) (41,161) (43,176) (45,162)
(47,207) (50,228) (52,217) (54,148) (63,191) (72,233) (127,206) (137,222) (154,218) (163,292)
(164,255) (170,269) (186,272) (188,293) (198,290) (199,258)

- u=150:

(185,248) (193,282) (64,234) (71,215) (65,121) (73,242) (78,231) (91,174) (20,75) (21,67)
(22,82) (29,160) (87,260) (112,238) (114,167) (94,184) (126,283) (142,289) (117,195) (124,189)
(129,229) (81,235) (86,224) (155,249) (163,276) (162,297) (187,263) (89,166) (68,109) (93,165)
(54,236) (76,188) (156,226) (136,244) (149,252) (95,220) (98,221) (59,210) (139,197) (152,225)
(96,264) (138,287) (123,257) (169,271) (77,255) (131,295) (84,205) (79,159) (150,232) (154,293)
(168,239) (173,230) (74,240) (111,267) (122,237) (172,301) (144,286) (100,250) (25,104) (26,118)
(141,209) (130,241) (103,262) (56,201) (127,186) (137,299) (153,285) (171,278) (179,253) (158,274)
(147,284) (47,218) (49,190) (119,200) (125,261) (105,268) (107,265) (135,254) (32,211) (33,207)
(99,183) (106,204) (108,251) (182,291) (101,247) (97,273) (116,178) (132,280) (133,288) (115,275)
(151,246) (30,140) (31,145) (191,296) (212,298) (213,277) (102,222) (177,294) (52,233) (58,198)
(157,245) (164,292) (170,269) (175,279) (1,34) (2,36) (3,38) (4,40) (5,42) (6,44)
(7,46) (8,48) (9,51) (10,53) (11,55) (12,57) (13,61) (14,63) (15,62) (16,66)
(17,69) (18,72) (19,70) (23,92) (24,90) (27,120) (28,180) (35,196) (37,202) (39,214)
(41,208) (43,223) (45,217) (50,227) (80,176) (83,216) (85,146) (88,194) (110,228) (113,243)
(128,219) (134,256) (143,270) (148,272) (161,258) (181,266) (192,259) (199,300) (203,290) (206,281)

- u=154:

(185,248) (193,282) (64,234) (71,215) (65,121) (73,242) (78,231) (91,174) (20,75) (21,67)
(22,82) (29,160) (87,260) (112,238) (114,167) (94,184) (126,283) (142,289) (117,195) (124,189)
(129,229) (81,235) (86,224) (155,249) (163,276) (162,297) (187,263) (89,166) (68,109) (93,165)
(54,236) (181,303) (156,226) (136,244) (149,252) (95,220) (98,221) (59,210) (139,197) (152,225)
(96,264) (138,287) (123,257) (169,271) (77,255) (146,279) (148,223) (79,159) (150,232) (154,293)
(168,239) (173,230) (74,240) (111,267) (122,237) (172,301) (144,286) (100,250) (25,104) (26,118)
(141,209) (130,241) (103,262) (175,302) (127,186) (137,299) (153,285) (171,278) (179,253) (158,274)
(147,284) (47,218) (49,190) (119,200) (125,261) (105,268) (107,265) (135,254) (32,211) (33,207)
(99,183) (106,204) (108,251) (182,291) (101,247) (97,273) (116,178) (132,280) (133,288) (115,275)
(151,246) (30,140) (31,145) (191,296) (212,298) (213,277) (102,222) (177,294) (52,233) (58,198)
(157,245) (164,292) (194,281) (199,290) (88,272) (192,258) (1,34) (2,36) (3,38) (4,40)
(5,42) (6,44) (7,46) (8,48) (9,51) (10,53) (11,55) (12,57) (13,60) (14,63)
(15,66) (16,70) (17,69) (18,170) (19,80) (23,92) (24,188) (27,128) (28,214) (35,131)
(37,202) (39,143) (41,216) (43,110) (45,217) (50,227) (56,180) (61,228) (72,120) (76,256)
(83,266) (84,205) (85,270) (90,208) (113,243) (134,295) (161,306) (176,269) (196,308) (201,307)
(203,300) (206,305) (219,304) (259,309)

- u=158:

(185,248) (193,282) (76,188) (65,121) (73,242) (78,231) (91,174) (20,75) (21,67) (22,82)
(29,160) (87,260) (137,299) (153,305) (94,184) (126,283) (142,289) (117,195) (124,189) (129,229)
(81,235) (83,269) (155,249) (163,276) (162,297) (187,263) (89,166) (68,109) (93,165) (54,236)
(181,303) (156,226) (136,244) (143,308) (95,220) (98,221) (59,210) (139,197) (152,225) (96,264)
(138,287) (123,257) (169,271) (77,255) (146,279) (128,312) (79,159) (150,232) (154,293) (168,239)
(173,230) (74,240) (111,267) (122,237) (172,301) (144,286) (100,250) (25,104) (26,118) (141,209)
(130,241) (103,262) (175,302) (127,186) (131,314) (134,311) (171,278) (179,253) (158,274) (147,284)
(47,218) (49,190) (119,200) (125,261) (105,268) (107,265) (135,254) (32,211) (33,207) (99,183)
(106,204) (108,251) (182,291) (101,247) (97,273) (116,178) (132,280) (133,288) (115,275) (151,246)
(30,140) (31,145) (191,296) (212,298) (213,277) (102,222) (177,294) (52,233) (58,198) (37,217)
(164,292) (194,281) (199,290) (196,317) (192,258) (201,270) (114,304) (113,285) (61,167) (85,272)
(1,34) (2,36) (3,38) (4,40) (5,42) (6,44) (7,46) (8,48) (9,51) (10,53)
(11,55) (12,57) (13,60) (14,62) (15,66) (16,69) (17,71) (18,203) (19,149) (23,72)
(24,112) (27,88) (28,80) (35,180) (39,227) (41,205) (43,161) (45,206) (50,176) (56,157)
(63,252) (70,214) (84,259) (86,256) (90,228) (92,224) (110,234) (120,223) (148,315) (170,266)
(202,306) (208,307) (215,300) (216,309) (219,316) (238,313) (243,310) (245,295)

- u=162:

(185,248) (193,282) (164,292) (176,321) (73,242) (78,231) (91,174) (20,75) (21,67) (22,82)
(29,160) (87,260) (137,299) (153,305) (94,184) (126,283) (142,289) (117,195) (124,189) (129,229)
(81,235) (83,269) (155,249) (163,276) (162,297) (187,263) (89,166) (68,109) (93,165) (54,236)
(181,303) (156,226) (136,244) (143,308) (95,220) (98,221) (59,210) (139,197) (152,225) (96,264)
(138,287) (123,257) (169,271) (77,255) (146,279) (128,312) (110,214) (141,209) (154,293) (168,239)
(173,230) (74,240) (111,267) (122,237) (172,301) (144,286) (100,250) (25,104) (116,178) (132,280)
(130,241) (103,262) (175,302) (127,186) (131,314) (134,311) (171,278) (179,253) (158,274) (147,284)
(47,218) (49,190) (149,316) (125,261) (105,268) (107,265) (135,254) (32,211) (33,207) (99,183)
(106,204) (108,251) (182,291) (101,247) (97,273) (119,245) (161,322) (133,288) (115,275) (151,246)
(30,140) (31,145) (191,296) (212,298) (213,277) (102,222) (177,294) (52,233) (58,198) (37,217)
(196,317) (56,180) (61,148) (234,315) (192,258) (201,270) (114,304) (113,285) (208,307) (216,313)
(85,272) (112,200) (118,310) (205,323) (223,324) (1,34) (2,36) (3,38) (4,40) (5,42)
(6,44) (7,46) (8,48) (9,51) (10,53) (11,55) (12,57) (13,60) (14,62) (15,64)
(16,69) (17,84) (18,70) (19,80) (23,79) (24,188) (26,76) (27,157) (28,203) (35,167)
(39,90) (41,121) (43,228) (45,238) (50,194) (63,159) (65,256) (71,259) (72,266) (86,224)
(88,170) (92,281) (120,232) (150,320) (199,290) (202,295) (206,309) (215,300) (219,325) (227,319)
(243,318) (252,306)

- u=166:

(256,331) (193,282) (164,292) (92,211) (73,242) (78,231) (91,174) (20,75) (21,67) (22,82)
(29,160) (87,260) (137,299) (153,305) (94,184) (126,283) (142,289) (117,195) (124,189) (129,229)
(81,235) (83,269) (155,249) (163,276) (162,297) (187,263) (89,166) (93,165) (54,236) (181,303)
(156,226) (136,244) (115,310) (95,220) (98,221) (59,210) (139,197) (152,225) (96,264) (138,287)
(123,257) (169,271) (77,255) (146,279) (228,295) (110,214) (141,209) (154,293) (168,239) (173,230)
(176,321) (111,267) (122,237) (172,301) (144,286) (100,250) (25,104) (116,178) (132,280) (130,241)
(103,262) (175,302) (127,186) (63,238) (134,311) (171,278) (179,253) (158,274) (147,284) (47,218)
(49,190) (149,316) (125,261) (105,268) (107,265) (150,320) (167,288) (33,207) (99,183) (106,204)
(108,251) (182,291) (101,247) (97,273) (119,245) (161,322) (215,327) (227,318) (151,246) (30,140)
(203,333) (206,309) (212,298) (213,277) (159,252) (177,294) (52,233) (58,198) (121,319) (145,330)
(56,180) (61,148) (234,315) (192,258) (201,270) (114,304) (113,285) (208,307) (216,313) (219,325)
(223,324) (224,306) (191,329) (194,326) (79,199) (31,85) (1,34) (2,36) (3,38) (4,40)
(5,42) (6,44) (7,46) (8,48) (9,50) (10,53) (11,55) (12,57) (13,60) (14,62)
(15,64) (16,66) (17,69) (18,71) (19,70) (23,84) (24,80) (26,118) (27,90) (28,222)
(32,74) (35,120) (37,217) (39,232) (41,196) (43,240) (45,205) (51,243) (65,170) (72,259)
(76,272) (86,200) (88,254) (102,266) (109,300) (112,296) (128,317) (131,314) (133,312) (135,323)
(143,308) (157,275) (185,281) (188,332) (202,290) (248,328)

- u=170:

(256,331) (193,282) (164,292) (92,211) (73,242) (78,231) (91,174) (20,75) (21,67) (22,82)
(29,160) (87,260) (137,299) (153,305) (94,184) (126,283) (142,289) (157,341) (166,338) (129,229)
(81,235) (83,269) (155,249) (163,276) (162,297) (187,263) (135,335) (93,165) (54,236) (181,303)
(156,226) (136,244) (115,310) (95,220) (98,221) (59,210) (139,197) (113,205) (96,264) (138,287)
(123,257) (169,271) (77,255) (146,279) (228,295) (110,214) (141,209) (154,293) (168,239) (173,230)
(176,321) (111,267) (122,237) (172,301) (144,286) (100,250) (23,65) (116,178) (132,280) (130,241)
(103,262) (175,302) (127,186) (63,238) (134,311) (171,278) (179,253) (158,274) (147,284) (47,218)
(49,190) (149,316) (125,261) (105,268) (107,265) (150,320) (167,288) (33,207) (99,183) (106,204)
(108,251) (182,291) (101,247) (97,273) (119,245) (161,322) (215,327) (227,318) (151,246) (30,140)
(203,333) (206,309) (212,298) (213,277) (159,252) (177,294) (52,233) (58,198) (121,319) (145,330)
(56,180) (61,148) (234,315) (192,258) (201,270) (114,304) (31,85) (208,307) (216,313) (219,325)
(223,324) (224,306) (191,329) (194,326) (79,199) (128,317) (131,323) (196,314) (133,334) (39,118)
(188,332) (1,34) (2,36) (3,38) (4,40) (5,42) (6,44) (7,46) (8,48) (9,50)
(10,53) (11,55) (12,57) (13,60) (14,62) (15,64) (16,66) (17,68) (18,74) (19,202)
(24,217) (25,189) (26,89) (27,88) (28,80) (32,112) (35,222) (37,90) (41,243) (43,120)
(45,225) (51,124) (69,248) (71,185) (72,266) (76,272) (84,281) (86,285) (102,290) (104,259)
(109,300) (117,195) (143,308) (152,312) (170,336) (200,296) (232,337) (240,328) (254,339) (275,340)

- u=174:

(170,349) (204,322) (212,298) (225,339) (234,315) (290,344) (254,331) (112,314) (118,200) (124,215)
(73,242) (78,231) (91,174) (20,75) (120,313) (164,292) (29,160) (143,340) (266,346) (153,305)
(94,184) (126,283) (142,289) (137,299) (166,338) (129,229) (81,235) (83,269) (155,249) (163,276)
(162,297) (187,263) (99,183) (93,165) (54,236) (181,303) (156,226) (136,244) (217,312) (95,220)
(98,221) (59,210) (114,304) (113,205) (96,264) (138,287) (123,257) (169,271) (77,255) (146,279)
(228,295) (105,285) (109,296) (154,293) (47,115) (84,245) (86,250) (87,213) (122,237) (172,301)
(144,286) (193,282) (23,65) (116,178) (132,280) (130,241) (103,262) (175,302) (127,186) (63,238)
(134,311) (171,278) (179,253) (158,274) (147,284) (188,348) (49,190) (149,316) (151,347) (152,343)
(107,265) (150,320) (167,288) (33,207) (110,214) (92,211) (108,251) (182,291) (101,247) (97,273)
(69,268) (37,240) (41,196) (43,243) (248,345) (202,275) (203,333) (206,309) (39,189) (159,252)
(177,294) (52,233) (76,281) (121,319) (145,330) (56,180) (61,148) (31,232) (32,197) (201,270)
(74,218) (246,342) (208,307) (71,259) (219,325) (51,139) (58,198) (191,329) (194,326) (79,199)
(104,310) (131,323) (141,336) (21,67) (125,223) (128,317) (135,308) (140,306) (161,324) (1,34)
(2,36) (3,38) (4,40) (5,42) (6,44) (7,46) (8,48) (9,50) (10,53) (11,55)
(12,57) (13,60) (14,62) (15,64) (16,66) (17,68) (18,119) (19,80) (22,216) (24,90)

(25,209) (26,230) (27,85) (28,106) (30,82) (35,88) (45,102) (70,133) (89,272) (100,256)
(111,176) (117,227) (157,328) (168,224) (173,258) (185,321) (192,337) (195,300) (222,334) (239,318)
(260,335) (261,332) (267,327) (277,341)

• u=178:

(170,349) (204,322) (212,298) (225,339) (234,315) (290,344) (111,318) (112,314) (118,200) (124,215)
(73,242) (78,231) (91,174) (20,75) (120,313) (164,292) (29,160) (143,340) (266,346) (153,305)
(94,184) (126,283) (142,289) (137,299) (166,338) (129,229) (81,235) (83,269) (155,249) (163,276)
(162,297) (187,263) (99,183) (93,165) (54,236) (181,303) (156,226) (136,244) (217,312) (95,220)
(98,221) (59,210) (114,304) (113,205) (96,264) (117,321) (119,328) (169,271) (77,255) (146,279)
(228,295) (105,285) (109,296) (154,293) (47,115) (84,245) (86,250) (133,316) (149,254) (172,301)
(144,286) (193,282) (23,65) (116,178) (132,280) (130,241) (261,332) (267,352) (127,186) (63,238)
(134,311) (171,278) (179,253) (158,274) (147,284) (188,348) (49,190) (151,347) (152,343) (107,265)
(150,320) (167,288) (33,207) (110,214) (92,211) (108,251) (182,291) (101,247) (97,273) (69,268)
(37,240) (41,196) (43,243) (248,345) (202,275) (203,333) (206,309) (39,189) (159,252) (177,294)
(52,233) (76,281) (121,319) (145,330) (56,180) (61,148) (31,232) (32,197) (201,270) (246,342)
(208,307) (71,259) (219,325) (51,139) (58,198) (191,329) (194,326) (79,199) (104,310) (131,323)
(141,336) (21,67) (125,223) (128,317) (135,308) (140,306) (161,227) (168,327) (1,34) (2,36)
(3,38) (4,40) (5,42) (6,44) (7,46) (8,48) (9,50) (10,53) (11,55) (12,57)
(13,60) (14,62) (15,64) (16,66) (17,68) (18,70) (19,72) (22,80) (24,173) (25,89)
(26,82) (27,87) (28,222) (30,175) (35,100) (45,106) (85,195) (88,272) (90,300) (102,258)
(103,230) (122,237) (123,331) (138,213) (157,324) (176,302) (185,356) (192,355) (209,353) (216,350)
(218,354) (224,287) (239,351) (256,357) (257,335) (260,337) (262,341) (277,334)

• u=182:

(170,349) (204,322) (212,298) (225,339) (234,315) (290,344) (111,318) (112,314) (118,200) (124,215)
(73,242) (78,231) (91,174) (20,75) (120,313) (164,292) (29,160) (143,340) (266,346) (153,305)
(94,184) (126,283) (142,289) (137,299) (166,338) (129,229) (81,235) (83,269) (155,249) (163,276)
(162,297) (187,263) (260,324) (278,353) (54,236) (181,303) (156,226) (227,354) (217,312) (95,220)
(98,221) (59,210) (114,304) (113,205) (96,264) (117,321) (119,328) (169,271) (77,255) (146,279)
(228,295) (105,285) (109,296) (154,293) (47,115) (84,245) (86,250) (133,316) (149,254) (172,301)
(144,286) (193,282) (23,65) (116,178) (132,280) (130,241) (261,332) (267,352) (127,186) (63,238)
(134,311) (123,331) (157,362) (158,274) (147,284) (188,348) (49,190) (151,347) (152,343) (107,265)
(150,320) (167,288) (33,207) (110,214) (92,211) (108,251) (182,291) (101,247) (97,273) (69,268)
(37,240) (41,196) (43,243) (248,345) (202,275) (203,333) (206,309) (39,189) (159,252) (177,294)
(52,233) (121,319) (145,330) (56,180) (61,148) (31,232) (32,197) (201,270) (246,342) (208,307)
(71,259) (219,325) (51,139) (58,198) (191,329) (194,326) (79,199) (104,310) (131,323) (141,336)
(21,67) (125,223) (128,317) (135,308) (140,306) (213,357) (216,272) (218,363) (161,355) (168,327)
(173,239) (1,34) (2,36) (3,38) (4,40) (5,42) (6,44) (7,46) (8,48) (9,50)
(10,53) (11,55) (12,57) (13,60) (14,62) (15,64) (16,66) (17,68) (18,70) (19,72)
(22,80) (24,85) (25,82) (26,89) (27,237) (28,102) (30,179) (35,100) (45,256) (74,175)
(87,171) (88,224) (90,257) (93,165) (99,176) (103,287) (106,277) (122,334) (136,350) (138,351)
(183,262) (185,341) (192,302) (195,358) (209,335) (222,337) (230,364) (244,356) (253,361) (258,365)
(281,359) (300,360)

• u=186:

(170,349) (204,322) (212,298) (225,339) (234,315) (290,344) (111,318) (112,314) (118,200) (124,215)
(73,242) (91,174) (20,75) (120,313) (164,292) (29,160) (143,340) (266,346) (165,277) (179,258)
(185,369) (195,366) (142,289) (137,299) (166,338) (129,229) (81,235) (83,269) (155,249) (163,276)
(162,297) (187,263) (260,324) (278,353) (54,236) (181,303) (156,226) (227,354) (217,312) (95,220)
(98,221) (59,210) (114,304) (113,205) (96,264) (117,321) (119,328) (169,271) (77,255) (146,279)
(228,295) (105,285) (109,296) (154,293) (47,115) (84,245) (86,250) (133,316) (149,254) (172,301)
(144,286) (193,282) (153,370) (192,359) (209,361) (130,241) (261,332) (267,352) (127,186) (63,238)
(19,65) (21,74) (157,362) (158,274) (147,284) (103,175) (106,171) (49,190) (110,188) (152,343)
(107,265) (150,320) (167,288) (33,207) (281,371) (287,364) (108,251) (182,291) (101,247) (97,273)
(69,268) (37,240) (41,196) (43,243) (248,345) (202,275) (203,333) (206,309) (39,189) (159,252)
(177,294) (52,233) (121,319) (145,330) (56,180) (61,148) (31,232) (32,197) (201,270) (246,342)
(208,307) (71,259) (219,325) (51,139) (58,198) (191,329) (194,326) (79,199) (104,310) (131,323)
(141,336) (231,350) (237,373) (257,365) (262,372) (230,331) (302,358) (213,357) (173,239) (178,341)
(161,355) (116,223) (122,334) (125,335) (132,280) (151,347) (1,34) (2,36) (3,38) (4,40)
(5,42) (6,44) (7,46) (8,48) (9,50) (10,53) (11,55) (12,57) (13,60) (14,62)
(15,64) (16,66) (17,68) (18,70) (22,80) (23,138) (24,85) (25,99) (26,183) (27,89)
(28,184) (30,128) (35,224) (45,218) (67,244) (72,176) (76,136) (82,216) (87,300) (88,214)
(90,256) (92,134) (93,311) (94,308) (100,253) (102,317) (123,283) (126,337) (135,351) (140,348)
(168,327) (211,360) (222,367) (272,356) (305,368) (306,363)

• u=190:

(28,126) (204,322) (212,298) (225,339) (234,315) (290,344) (306,363) (112,314) (118,200) (124,215)
(73,242) (91,174) (20,75) (120,313) (164,292) (29,160) (92,176) (266,346) (165,277) (179,258)
(185,369) (195,366) (142,289) (137,299) (166,338) (155,367) (157,376) (114,334) (117,321) (119,337)
(122,327) (187,263) (260,324) (278,353) (54,236) (181,303) (156,226) (227,354) (217,312) (95,220)

(98,221) (59,210) (256,360) (113,205) (96,264) (78,244) (81,235) (169,271) (77,255) (146,279)
(228,295) (105,285) (109,296) (154,293) (47,115) (84,245) (86,250) (25,67) (149,254) (123,272)
(144,286) (193,282) (153,370) (192,359) (209,361) (130,241) (261,332) (267,352) (127,186) (63,238)
(19,65) (21,74) (11,55) (158,274) (147,284) (103,175) (106,171) (49,190) (110,188) (152,343)
(107,265) (150,320) (167,288) (33,207) (281,371) (287,364) (108,251) (182,291) (101,247) (97,273)
(69,268) (37,240) (41,196) (43,243) (248,345) (202,275) (203,333) (206,309) (39,189) (159,252)
(177,294) (52,233) (121,319) (145,330) (56,180) (61,148) (31,232) (32,197) (201,270) (246,342)
(208,307) (71,259) (219,325) (51,139) (58,198) (191,329) (194,326) (229,374) (253,379) (131,323)
(141,336) (231,350) (237,373) (257,365) (262,372) (230,331) (302,358) (213,357) (173,239) (178,341)
(161,355) (116,223) (23,85) (125,335) (132,280) (151,347) (184,381) (211,368) (216,375) (1,34)
(2,36) (3,38) (4,40) (5,42) (6,44) (7,46) (8,48) (9,50) (10,53) (12,57)
(13,60) (14,62) (15,64) (16,66) (17,68) (18,70) (22,82) (24,87) (26,100) (27,88)
(30,143) (35,135) (45,222) (72,283) (76,170) (79,199) (83,218) (89,311) (90,269) (93,276)
(94,308) (99,305) (102,310) (104,317) (111,297) (128,349) (129,318) (133,340) (134,249) (136,351)
(138,328) (140,300) (162,378) (163,316) (168,377) (172,301) (183,356) (214,348) (224,380) (304,362)

- u=194:

(31,151) (33,162) (35,129) (234,315) (290,344) (306,363) (112,314) (118,200) (124,215) (73,242)
(91,174) (20,75) (120,313) (164,292) (29,160) (92,176) (266,346) (165,277) (15,64) (16,66)
(17,68) (18,70) (137,299) (166,338) (155,367) (89,310) (94,318) (104,304) (133,316) (122,327)
(187,263) (260,324) (278,353) (79,269) (181,303) (156,226) (227,354) (217,312) (95,220) (98,221)
(59,210) (256,360) (113,205) (96,264) (78,244) (81,235) (169,271) (77,255) (146,279) (228,295)
(105,285) (109,296) (154,293) (47,115) (84,245) (86,250) (25,67) (149,254) (123,272) (144,286)
(193,282) (93,311) (192,359) (209,361) (130,241) (261,332) (267,352) (127,186) (63,238) (19,65)
(21,74) (11,55) (138,349) (147,284) (103,175) (106,171) (49,190) (110,188) (152,343) (107,265)
(150,320) (167,288) (276,389) (281,371) (287,364) (153,301) (157,376) (170,387) (172,375) (183,369)
(184,380) (213,366) (283,383) (248,345) (202,275) (203,333) (206,309) (39,189) (159,252) (177,294)
(52,233) (99,321) (102,317) (56,180) (61,148) (243,377) (32,197) (201,270) (246,342) (208,307)
(71,259) (219,325) (51,139) (58,198) (191,329) (194,326) (229,374) (253,379) (131,323) (141,336)
(231,350) (237,373) (257,365) (262,372) (230,331) (302,358) (111,308) (173,239) (178,341) (161,355)
(116,223) (23,85) (218,378) (224,322) (232,348) (27,88) (28,114) (30,207) (37,222) (199,388)
(145,370) (163,298) (211,385) (212,330) (185,300) (135,249) (136,337) (140,356) (168,382) (1,34)
(2,36) (3,38) (4,40) (5,42) (6,44) (7,46) (8,48) (9,50) (10,53) (12,57)
(13,76) (14,62) (22,80) (24,179) (26,182) (41,225) (43,117) (45,204) (54,236) (60,258)
(69,273) (72,216) (83,289) (87,196) (90,297) (97,268) (100,247) (101,280) (108,251) (119,328)
(121,347) (125,335) (126,339) (128,351) (132,340) (134,291) (142,362) (143,319) (158,357) (195,368)
(214,274) (240,386) (305,384) (334,381)

- u=198:

(57,230) (283,383) (72,151) (234,315) (290,344) (83,240) (90,274) (93,311) (124,215) (73,242)
(91,174) (13,60) (120,313) (164,292) (29,160) (92,176) (266,346) (145,343) (152,368) (16,66)
(17,68) (18,70) (137,299) (166,338) (155,367) (89,310) (94,318) (142,216) (133,316) (122,327)
(187,263) (260,324) (278,353) (79,269) (181,303) (156,226) (227,354) (217,312) (95,220) (98,221)
(59,210) (256,360) (113,205) (96,264) (78,244) (81,235) (87,297) (77,255) (146,279) (228,295)
(105,285) (109,296) (154,293) (47,115) (86,250) (119,326) (121,330) (262,382) (144,286) (193,282)
(132,291) (192,359) (209,361) (130,241) (261,332) (267,352) (127,186) (63,238) (19,65) (21,74)
(11,55) (323,384) (147,284) (103,175) (106,171) (101,213) (110,188) (27,136) (107,265) (150,320)
(167,288) (276,389) (281,371) (287,364) (153,301) (125,304) (170,387) (172,375) (183,369) (129,258)
(97,305) (102,317) (108,337) (248,345) (202,275) (203,333) (206,309) (39,189) (159,252) (177,294)
(52,233) (99,321) (280,340) (56,180) (61,148) (243,377) (32,197) (201,270) (246,342) (208,307)
(71,259) (219,325) (51,139) (58,198) (191,329) (196,395) (229,374) (253,379) (35,117) (185,386)
(231,350) (237,373) (257,365) (195,351) (214,390) (302,358) (225,339) (245,391) (178,341) (161,355)
(116,223) (23,85) (218,378) (224,322) (232,348) (247,362) (28,114) (30,207) (37,222) (199,388)
(254,372) (163,298) (211,385) (140,370) (149,331) (162,366) (158,381) (179,334) (306,363) (143,356)
(111,336) (123,319) (126,184) (26,131) (31,141) (1,34) (2,36) (3,38) (4,40) (5,42)
(6,44) (7,46) (8,48) (9,50) (10,53) (12,54) (14,69) (15,64) (20,212) (22,165)
(24,173) (25,251) (33,134) (41,104) (43,190) (45,272) (49,277) (62,268) (67,169) (75,236)
(76,271) (80,128) (82,273) (88,308) (100,194) (112,314) (118,289) (135,335) (138,349) (157,376)
(168,300) (182,396) (200,397) (204,357) (239,380) (249,393) (328,394) (347,392)

- u=202:

(57,230) (283,383) (72,151) (234,315) (290,344) (83,240) (90,274) (93,311) (124,215) (73,242)
(91,174) (13,60) (157,376) (165,397) (29,160) (92,176) (266,346) (145,343) (152,368) (16,66)
(17,68) (18,70) (137,299) (166,338) (155,367) (89,310) (94,318) (142,216) (133,316) (122,327)
(187,263) (260,324) (278,353) (79,269) (181,303) (156,226) (227,354) (217,312) (95,220) (98,221)
(59,210) (256,360) (113,205) (96,264) (78,244) (81,235) (87,297) (77,255) (146,279) (228,295)
(105,285) (109,296) (154,293) (47,115) (119,326) (121,330) (262,382) (144,286) (193,282) (132,291)
(192,359) (209,361) (130,241) (261,332) (267,352) (127,186) (63,238) (19,65) (21,74) (11,55)
(323,384) (147,284) (103,175) (106,171) (101,213) (110,188) (27,136) (107,265) (150,320) (167,288)
(276,389) (281,371) (287,364) (153,301) (125,304) (170,387) (172,375) (183,369) (129,258) (97,305)
(102,317) (108,337) (248,345) (202,275) (203,333) (206,309) (39,189) (159,252) (177,294) (52,233)
(99,321) (280,340) (56,180) (61,148) (243,377) (32,197) (201,270) (246,342) (208,307) (71,259)
(219,325) (51,139) (58,198) (239,400) (250,393) (251,398) (253,379) (35,117) (185,386) (231,350)

(237,373) (257,365) (195,351) (214,390) (302,358) (225,339) (245,391) (178,341) (161,355) (116,223)
(23,85) (218,378) (224,322) (232,348) (247,362) (33,134) (41,182) (37,222) (199,388) (254,372)
(163,298) (211,385) (140,370) (149,331) (162,366) (158,381) (179,334) (306,363) (143,356) (111,336)
(123,319) (173,399) (191,329) (1,34) (2,36) (3,38) (4,40) (5,42) (6,44) (7,46)
(8,48) (9,50) (10,53) (12,54) (14,62) (15,64) (20,75) (22,88) (24,126) (25,196)
(26,84) (28,138) (30,207) (31,76) (43,271) (45,272) (49,112) (67,300) (69,268) (80,212)
(82,168) (100,249) (104,335) (114,314) (118,313) (120,273) (128,292) (131,328) (135,349) (141,347)
(164,357) (169,403) (184,404) (190,401) (194,396) (200,392) (204,395) (229,374) (236,380) (277,405)
(289,394) (308,402)

- u=206:

(131,322) (283,383) (72,151) (234,315) (290,344) (57,209) (86,316) (102,196) (124,215) (49,272)
(80,313) (13,60) (157,376) (165,397) (29,160) (92,176) (266,346) (145,343) (152,368) (16,66)
(17,68) (18,70) (137,299) (166,338) (155,367) (89,310) (94,318) (142,216) (150,387) (153,301)
(187,263) (260,324) (278,353) (79,269) (181,303) (156,226) (227,354) (217,312) (161,355) (163,394)
(59,210) (256,360) (113,205) (96,264) (78,244) (81,235) (87,297) (77,255) (146,279) (228,295)
(105,285) (109,296) (154,293) (47,115) (119,326) (121,330) (262,382) (144,286) (193,282) (132,291)
(25,135) (26,112) (130,241) (261,332) (267,352) (127,186) (63,238) (19,65) (21,74) (11,55)
(323,384) (147,284) (103,175) (106,171) (101,213) (110,188) (27,136) (107,265) (28,172) (30,128)
(276,389) (281,371) (287,364) (179,406) (236,413) (242,411) (167,288) (183,369) (129,258) (308,410)
(97,305) (100,328) (248,345) (202,275) (91,174) (93,311) (39,189) (159,252) (177,294) (52,233)
(99,321) (280,340) (56,180) (61,148) (243,377) (32,197) (201,270) (246,342) (208,307) (71,259)
(219,325) (51,139) (58,198) (239,400) (250,393) (251,398) (253,379) (35,117) (185,386) (231,350)
(237,373) (257,365) (195,351) (214,390) (302,358) (225,339) (173,407) (184,300) (190,392) (191,402)
(23,85) (218,378) (104,249) (271,403) (247,362) (33,134) (41,182) (37,222) (199,388) (254,372)
(224,327) (273,401) (73,203) (84,277) (162,366) (69,289) (76,268) (306,363) (143,356) (111,336)
(123,319) (141,347) (149,331) (314,380) (349,412) (229,334) (232,381) (1,34) (2,36) (3,38)
(4,40) (5,42) (6,44) (7,46) (8,48) (9,50) (10,53) (12,54) (14,62) (15,64)
(20,75) (22,168) (24,82) (31,169) (43,178) (45,212) (67,192) (83,298) (90,304) (95,274)
(98,221) (108,337) (114,309) (116,333) (118,292) (120,320) (122,348) (125,361) (126,329) (133,240)
(138,335) (140,375) (158,341) (164,317) (170,408) (194,399) (200,357) (204,374) (206,405) (207,370)
(211,395) (220,391) (223,396) (230,385) (245,409) (359,404)

- u=210:

(133,300) (140,337) (143,385) (234,315) (290,344) (57,209) (86,316) (102,196) (124,215) (49,272)
(80,313) (13,60) (157,376) (165,397) (29,160) (92,176) (266,346) (145,343) (152,368) (16,66)
(17,68) (18,70) (137,299) (166,338) (155,367) (89,310) (94,318) (142,216) (111,326) (114,336)
(187,263) (260,324) (278,353) (79,269) (181,303) (156,226) (227,354) (217,312) (223,396) (163,394)
(59,210) (256,360) (113,205) (96,264) (78,244) (81,235) (87,297) (77,255) (146,279) (228,295)
(105,285) (109,296) (154,293) (47,115) (161,355) (164,381) (262,382) (144,286) (193,282) (132,291)
(25,135) (26,112) (130,241) (261,332) (267,352) (127,186) (63,238) (19,65) (21,74) (11,55)
(323,384) (147,284) (103,175) (106,171) (101,213) (110,188) (27,136) (107,265) (28,172) (30,128)
(276,389) (281,371) (287,364) (179,406) (207,421) (211,420) (167,288) (183,369) (129,258) (308,410)
(97,305) (100,328) (248,345) (202,275) (91,174) (93,311) (39,189) (236,413) (242,411) (52,233)
(88,317) (280,340) (56,180) (61,148) (243,377) (32,197) (201,270) (246,342) (208,307) (71,259)
(219,325) (51,139) (58,198) (239,400) (250,393) (251,398) (253,379) (35,117) (185,386) (231,350)
(237,373) (257,365) (121,359) (122,335) (302,358) (225,339) (173,407) (125,304) (190,392) (191,402)
(23,85) (218,378) (104,249) (271,403) (247,362) (33,134) (41,182) (37,222) (199,388) (254,372)
(224,327) (273,401) (73,203) (84,277) (162,366) (69,289) (43,168) (306,363) (116,357) (118,292)
(123,319) (141,347) (149,331) (314,380) (349,412) (229,334) (220,419) (221,374) (230,414) (204,361)
(240,395) (245,391) (283,418) (309,416) (1,34) (2,36) (3,38) (4,40) (5,42) (6,44)
(7,46) (8,48) (9,50) (10,53) (12,54) (14,62) (15,64) (20,75) (22,170) (24,200)
(31,131) (45,194) (67,184) (72,298) (76,214) (82,321) (83,274) (95,330) (98,268) (99,192)
(108,153) (119,322) (120,320) (126,333) (138,294) (150,387) (151,356) (158,383) (159,399) (169,405)
(177,348) (178,341) (195,390) (206,370) (212,404) (232,415) (252,375) (301,417) (329,408) (351,409)

- u=214:

(158,404) (159,330) (143,385) (234,315) (290,344) (57,209) (86,316) (102,196) (124,215) (95,321)
(98,298) (13,60) (157,376) (165,397) (29,160) (266,346) (145,343) (152,368) (16,66) (17,68)
(18,70) (137,299) (166,338) (155,367) (89,310) (94,318) (142,216) (111,326) (114,336) (187,263)
(260,324) (278,353) (79,269) (181,303) (156,226) (227,354) (217,312) (223,396) (163,394) (59,210)
(256,360) (113,205) (96,264) (78,244) (81,235) (83,274) (77,255) (146,279) (228,295) (105,285)
(109,296) (154,293) (47,115) (141,347) (149,331) (262,382) (144,286) (193,282) (132,291) (25,135)
(26,112) (130,241) (261,332) (267,352) (127,186) (63,238) (19,65) (21,74) (11,55) (323,384)
(147,284) (103,175) (106,171) (101,213) (110,188) (27,136) (107,265) (28,172) (30,128) (276,389)
(281,371) (287,364) (178,381) (179,406) (211,420) (167,288) (49,184) (72,151) (308,410) (97,305)
(100,328) (248,345) (202,275) (91,174) (93,311) (39,189) (236,413) (242,411) (52,233) (88,317)
(280,340) (56,180) (61,148) (243,377) (32,197) (201,270) (246,342) (208,307) (71,259) (219,325)
(51,139) (58,198) (239,400) (250,393) (251,398) (253,379) (35,117) (185,386) (231,350) (237,373)
(257,365) (121,359) (122,335) (302,358) (225,339) (173,407) (194,399) (252,408) (268,417) (23,85)

(218,378)	(104,249)	(271,403)	(247,362)	(33,134)	(41,182)	(37,222)	(199,388)	(254,372)	(224,327)
(273,401)	(73,203)	(84,277)	(162,366)	(69,289)	(43,168)	(306,363)	(116,357)	(118,292)	(309,426)
(120,356)	(125,304)	(314,380)	(349,412)	(229,334)	(220,419)	(221,374)	(230,414)	(204,361)	(240,395)
(245,391)	(212,375)	(301,424)	(341,425)	(1,34)	(2,36)	(3,38)	(4,40)	(5,42)	(6,44)
(7,46)	(8,48)	(9,50)	(10,53)	(12,54)	(14,62)	(15,64)	(20,75)	(22,67)	(24,191)
(31,169)	(45,161)	(76,319)	(80,272)	(82,140)	(87,283)	(90,190)	(99,322)	(108,294)	(119,313)
(123,337)	(126,351)	(129,369)	(131,370)	(133,297)	(138,333)	(150,387)	(153,355)	(164,409)	(170,405)
(176,383)	(177,421)	(183,416)	(192,402)	(195,392)	(200,348)	(206,423)	(207,418)	(214,390)	(232,415)
(258,428)	(300,429)	(320,427)	(329,422)						

- u=218:

(205,360)	(208,307)	(143,385)	(271,403)	(313,431)	(57,209)	(86,316)	(102,196)	(124,215)	(95,321)
(98,298)	(13,60)	(157,376)	(165,397)	(29,160)	(266,346)	(272,425)	(297,421)	(16,66)	(22,67)
(24,82)	(137,299)	(130,322)	(133,370)	(89,310)	(142,216)	(111,326)	(114,336)	(187,263)	(260,324)
(278,353)	(79,269)	(181,303)	(156,226)	(227,354)	(217,312)	(223,396)	(163,394)	(59,210)	(191,329)
(192,369)	(96,264)	(78,244)	(81,235)	(83,274)	(77,255)	(146,279)	(228,295)	(105,285)	(109,296)
(154,293)	(47,115)	(141,347)	(149,331)	(262,382)	(123,320)	(126,337)	(132,291)	(25,135)	(26,112)
(319,430)	(183,433)	(267,352)	(127,186)	(63,238)	(19,65)	(21,74)	(11,55)	(92,253)	(99,342)
(103,175)	(106,171)	(101,213)	(110,188)	(27,136)	(107,265)	(28,172)	(30,128)	(276,389)	(281,371)
(287,364)	(178,381)	(232,434)	(254,418)	(167,288)	(49,184)	(72,151)	(308,410)	(97,305)	(100,328)
(248,345)	(202,275)	(91,174)	(93,311)	(39,189)	(153,402)	(161,355)	(52,233)	(88,317)	(280,340)
(58,200)	(76,286)	(243,377)	(32,197)	(201,270)	(242,414)	(249,419)	(259,435)	(323,415)	(61,230)
(71,164)	(195,404)	(198,405)	(251,398)	(206,420)	(35,117)	(185,386)	(231,350)	(237,373)	(257,365)
(121,359)	(122,335)	(302,358)	(225,339)	(290,344)	(194,399)	(252,408)	(268,417)	(23,85)	(218,378)
(113,348)	(119,343)	(247,362)	(33,134)	(41,182)	(37,222)	(199,388)	(300,429)	(224,327)	(168,284)
(73,203)	(84,277)	(283,390)	(69,289)	(80,176)	(87,158)	(306,363)	(116,357)	(118,292)	(309,426)
(241,436)	(250,393)	(314,380)	(349,412)	(229,334)	(261,428)	(120,356)	(125,304)	(204,361)	(214,318)
(245,391)	(212,375)	(301,424)	(170,387)	(177,423)	(193,282)	(207,432)	(219,325)	(246,372)	(190,330)
(211,409)	(220,416)	(221,437)	(1,34)	(2,36)	(3,38)	(4,40)	(5,42)	(6,44)	(7,46)
(8,48)	(9,50)	(10,147)	(12,239)	(14,75)	(15,64)	(17,68)	(18,70)	(20,104)	(31,179)
(43,131)	(45,173)	(51,138)	(53,108)	(54,258)	(56,240)	(62,162)	(90,338)	(129,341)	(139,383)
(140,374)	(144,367)	(145,384)	(148,395)	(150,333)	(152,351)	(155,400)	(159,392)	(166,406)	(169,294)
(180,366)	(234,422)	(236,407)	(256,401)	(273,315)	(332,413)	(368,411)	(379,427)		

- u=222:

(236,381)	(239,438)	(143,385)	(271,403)	(313,431)	(57,209)	(86,316)	(102,196)	(124,215)	(95,321)
(98,298)	(13,60)	(157,376)	(165,397)	(29,160)	(266,346)	(272,425)	(297,421)	(16,66)	(22,67)
(24,82)	(137,299)	(145,368)	(147,392)	(89,310)	(142,216)	(111,326)	(114,336)	(187,263)	(260,324)
(278,353)	(79,269)	(181,303)	(156,226)	(227,354)	(217,312)	(223,396)	(163,394)	(59,210)	(191,329)
(192,369)	(78,244)	(81,235)	(83,274)	(77,255)	(146,279)	(228,295)	(105,285)	(109,296)	(154,293)
(47,115)	(141,347)	(149,331)	(262,382)	(123,320)	(126,337)	(132,291)	(25,135)	(26,112)	(319,430)
(51,138)	(267,352)	(127,186)	(63,238)	(19,65)	(21,74)	(11,55)	(92,253)	(99,342)	(103,175)
(106,171)	(101,213)	(110,188)	(27,136)	(107,265)	(28,172)	(30,128)	(276,389)	(281,371)	(221,364)
(148,395)	(315,440)	(332,413)	(167,288)	(49,184)	(72,151)	(308,410)	(97,305)	(100,328)	(248,345)
(202,275)	(91,174)	(93,311)	(39,189)	(153,402)	(161,355)	(52,233)	(88,317)	(280,340)	(58,200)
(76,286)	(243,377)	(32,197)	(201,270)	(242,414)	(249,419)	(259,435)	(323,415)	(61,230)	(71,164)
(195,404)	(198,405)	(251,398)	(206,420)	(35,117)	(185,386)	(231,350)	(237,373)	(257,365)	(131,379)
(133,367)	(302,358)	(225,339)	(290,344)	(194,399)	(252,408)	(268,417)	(23,85)	(218,378)	(113,348)
(119,343)	(247,362)	(33,134)	(41,182)	(37,222)	(199,388)	(300,429)	(224,327)	(168,284)	(94,333)
(121,359)	(283,390)	(69,289)	(56,144)	(90,254)	(306,363)	(116,357)	(118,292)	(309,426)	(258,444)
(264,401)	(314,380)	(349,412)	(229,334)	(261,428)	(120,356)	(125,304)	(204,361)	(214,318)	(245,391)
(212,375)	(301,424)	(205,393)	(208,445)	(193,282)	(207,432)	(219,325)	(246,372)	(190,330)	(240,443)
(241,436)	(256,439)	(287,383)	(273,441)	(294,442)	(1,34)	(2,36)	(3,38)	(4,40)	(5,42)
(6,44)	(7,46)	(8,48)	(9,50)	(10,53)	(12,140)	(14,62)	(15,64)	(17,68)	(18,70)
(20,150)	(31,130)	(43,104)	(45,87)	(54,250)	(73,277)	(75,159)	(80,180)	(84,335)	(108,341)
(122,177)	(129,322)	(139,351)	(152,406)	(155,407)	(158,411)	(162,360)	(166,416)	(169,409)	(170,387)
(173,400)	(176,422)	(178,370)	(179,423)	(183,338)	(203,374)	(211,427)	(220,433)	(232,434)	(234,418)
(307,384)	(366,437)								

- u=226:

(236,381)	(239,438)	(143,385)	(271,403)	(313,431)	(57,209)	(86,316)	(102,196)	(124,215)	(61,230)
(71,164)	(157,376)	(165,397)	(29,160)	(266,346)	(272,425)	(297,421)	(16,66)	(22,67)	(24,82)
(137,299)	(178,378)	(179,406)	(89,310)	(142,216)	(111,326)	(114,336)	(187,263)	(260,324)	(278,353)
(79,269)	(181,303)	(156,226)	(227,354)	(217,312)	(223,396)	(163,394)	(191,395)	(192,416)	(159,411)
(78,244)	(81,235)	(83,274)	(77,255)	(146,279)	(228,295)	(43,104)	(45,129)	(154,293)	(47,115)
(161,374)	(162,356)	(262,382)	(123,320)	(126,337)	(132,291)	(25,135)	(26,112)	(319,430)	(51,138)
(267,352)	(127,186)	(63,238)	(19,65)	(21,74)	(11,55)	(92,253)	(99,342)	(103,175)	(248,345)
(256,451)	(259,436)	(27,136)	(107,265)	(28,172)	(30,128)	(276,389)	(281,371)	(294,442)	(307,437)
(315,440)	(332,413)	(167,288)	(49,184)	(72,151)	(308,410)	(97,305)	(218,441)	(229,400)	(240,443)
(91,174)	(93,311)	(39,189)	(122,277)	(125,341)	(52,233)	(88,317)	(280,340)	(58,200)	(76,286)
(243,377)	(32,197)	(201,270)	(249,419)	(250,418)	(273,433)	(323,415)	(84,338)	(87,158)	(195,404)
(198,405)	(251,398)	(206,420)	(35,117)	(185,386)	(231,350)	(237,373)	(257,365)	(131,379)	(190,370)
(202,275)	(241,439)	(242,414)	(285,384)	(322,450)	(347,452)	(23,85)	(152,402)	(176,434)	(177,423)

(247,362)	(33,134)	(41,182)	(37,222)	(199,388)	(300,429)	(224,327)	(168,284)	(94,333)	(121,359)
(283,390)	(69,289)	(56,144)	(90,254)	(306,363)	(116,357)	(118,292)	(309,426)	(258,444)	(264,401)
(314,380)	(349,412)	(169,422)	(170,427)	(130,366)	(139,331)	(204,361)	(214,318)	(245,391)	(212,375)
(301,424)	(205,393)	(208,445)	(193,282)	(207,432)	(219,325)	(246,372)	(203,448)	(221,364)	(232,449)
(234,446)	(287,383)	(261,428)	(302,358)	(304,453)	(153,335)	(166,417)	(173,369)	(1,34)	(2,36)
(3,38)	(4,40)	(5,42)	(6,44)	(7,46)	(8,48)	(9,50)	(10,53)	(12,54)	(13,60)
(14,62)	(15,64)	(17,68)	(18,70)	(20,225)	(31,96)	(59,210)	(73,328)	(75,213)	(80,180)
(95,351)	(100,155)	(101,348)	(105,298)	(106,339)	(108,220)	(109,344)	(110,188)	(113,296)	(119,368)
(120,360)	(133,367)	(140,194)	(141,343)	(145,329)	(147,334)	(148,392)	(149,355)	(150,290)	(171,399)
(183,409)	(211,387)	(252,408)	(268,447)	(321,435)	(330,407)				

- u=230:

(140,387)	(90,254)	(105,213)	(271,403)	(313,431)	(57,209)	(86,316)	(102,196)	(124,215)	(61,230)
(71,164)	(157,376)	(165,397)	(29,160)	(266,346)	(272,425)	(297,421)	(16,66)	(22,67)	(24,82)
(137,299)	(178,378)	(179,406)	(89,310)	(142,216)	(111,326)	(114,336)	(187,263)	(260,324)	(278,353)
(79,269)	(181,303)	(156,226)	(227,354)	(217,312)	(223,396)	(139,344)	(141,343)	(192,416)	(159,411)
(78,244)	(81,235)	(83,274)	(77,255)	(108,173)	(110,366)	(43,104)	(45,129)	(154,293)	(47,115)
(161,374)	(162,356)	(262,382)	(123,320)	(126,337)	(132,291)	(25,135)	(26,112)	(319,430)	(51,138)
(267,352)	(127,186)	(63,238)	(19,65)	(21,74)	(11,55)	(92,253)	(99,342)	(103,175)	(248,345)
(256,451)	(259,436)	(27,136)	(107,265)	(28,172)	(30,128)	(276,389)	(281,371)	(294,442)	(307,437)
(315,440)	(332,413)	(167,288)	(49,184)	(72,151)	(308,410)	(97,305)	(218,441)	(229,400)	(18,70)
(20,109)	(93,311)	(39,189)	(122,277)	(125,341)	(52,233)	(88,317)	(280,340)	(58,200)	(76,286)
(243,377)	(32,197)	(201,270)	(249,419)	(250,418)	(273,433)	(323,415)	(84,338)	(87,158)	(195,404)
(198,405)	(251,398)	(206,420)	(35,117)	(185,386)	(98,331)	(101,155)	(145,373)	(146,330)	(190,370)
(202,275)	(241,439)	(242,414)	(236,435)	(237,449)	(239,422)	(23,85)	(152,402)	(176,434)	(91,174)
(247,362)	(33,134)	(41,182)	(37,222)	(199,388)	(300,429)	(224,327)	(168,284)	(94,333)	(121,359)
(283,390)	(69,289)	(56,144)	(73,150)	(306,363)	(116,357)	(118,292)	(309,426)	(113,169)	(264,401)
(314,380)	(349,412)	(252,392)	(149,328)	(171,367)	(183,334)	(204,361)	(214,318)	(245,391)	(212,375)
(232,351)	(234,460)	(119,355)	(95,282)	(96,358)	(207,432)	(219,325)	(246,372)	(225,339)	(268,461)
(170,427)	(177,438)	(287,383)	(120,360)	(130,364)	(304,453)	(153,335)	(166,417)	(203,458)	(210,348)
(188,321)	(279,455)	(298,454)	(257,369)	(295,350)	(301,379)	(302,445)	(322,450)	(347,452)	(1,34)
(2,36)	(3,38)	(4,40)	(5,42)	(6,46)	(7,48)	(8,50)	(9,60)	(10,53)	(12,148)
(13,62)	(14,258)	(15,54)	(17,64)	(31,131)	(44,261)	(59,296)	(68,191)	(75,220)	(80,329)
(106,365)	(133,368)	(143,385)	(147,407)	(163,409)	(180,428)	(193,399)	(194,447)	(205,393)	(208,394)
(211,456)	(221,424)	(228,459)	(231,423)	(240,444)	(285,384)	(290,457)	(381,448)	(395,443)	(408,446)

- u=234:

(96,270)	(100,358)	(254,451)	(256,424)	(268,394)	(57,209)	(86,316)	(124,215)	(315,440)	(323,415)
(157,376)	(165,397)	(29,160)	(266,346)	(272,425)	(297,421)	(16,66)	(22,67)	(24,82)	(137,299)
(178,378)	(179,406)	(89,310)	(183,387)	(111,326)	(114,336)	(187,263)	(260,324)	(278,353)	(79,269)
(181,303)	(156,226)	(227,354)	(217,312)	(249,419)	(250,347)	(141,343)	(192,416)	(159,411)	(78,244)
(81,235)	(83,274)	(77,255)	(108,173)	(110,366)	(43,104)	(45,129)	(154,293)	(47,115)	(161,374)
(162,356)	(248,399)	(259,436)	(290,457)	(132,291)	(25,135)	(26,112)	(319,430)	(51,138)	(267,352)
(127,186)	(63,238)	(19,65)	(21,74)	(11,55)	(92,253)	(99,342)	(103,175)	(193,452)	(95,223)
(101,294)	(123,216)	(140,382)	(143,391)	(30,128)	(276,389)	(281,371)	(196,296)	(211,375)	(17,68)
(61,230)	(167,288)	(49,184)	(72,151)	(308,410)	(97,305)	(218,441)	(229,400)	(18,70)	(20,109)
(93,311)	(39,189)	(122,277)	(125,341)	(52,233)	(80,283)	(280,340)	(58,200)	(76,286)	(243,377)
(32,197)	(107,265)	(121,365)	(148,334)	(273,433)	(142,408)	(84,338)	(87,158)	(195,404)	(198,405)
(251,398)	(206,420)	(35,117)	(185,386)	(62,136)	(85,258)	(145,373)	(146,330)	(190,370)	(202,275)
(262,407)	(271,434)	(236,435)	(237,449)	(239,422)	(15,64)	(180,444)	(191,456)	(91,174)	(247,362)
(33,134)	(41,182)	(37,222)	(199,388)	(300,429)	(224,327)	(168,284)	(94,333)	(220,469)	(245,381)
(69,289)	(56,144)	(73,150)	(306,363)	(116,357)	(152,402)	(214,409)	(309,426)	(113,169)	(264,401)
(314,380)	(349,412)	(252,392)	(149,328)	(171,367)	(221,466)	(228,465)	(231,423)	(105,213)	(106,359)
(261,467)	(232,351)	(234,460)	(119,355)	(31,98)	(75,322)	(207,432)	(219,325)	(163,396)	(172,418)
(201,464)	(170,427)	(177,438)	(287,383)	(120,360)	(130,364)	(304,453)	(153,335)	(166,417)	(203,458)
(210,348)	(188,321)	(279,455)	(298,454)	(257,369)	(295,350)	(301,379)	(302,445)	(329,428)	(307,437)
(320,443)	(332,413)	(345,450)	(337,385)	(368,462)	(1,34)	(2,36)	(3,38)	(4,40)	(5,46)
(6,194)	(7,164)	(8,48)	(9,118)	(10,53)	(12,126)	(13,242)	(14,225)	(23,155)	(27,147)
(28,90)	(42,240)	(44,282)	(50,88)	(54,285)	(59,205)	(60,208)	(71,331)	(131,318)	(133,395)
(139,176)	(204,439)	(212,384)	(241,446)	(246,463)	(292,339)	(313,431)	(317,461)	(344,448)	(361,468)
(372,414)	(390,459)	(393,447)	(403,442)						

- u=238:

(96,270)	(100,358)	(254,451)	(237,448)	(249,461)	(57,209)	(86,316)	(124,215)	(315,440)	(323,415)
(157,376)	(165,397)	(29,160)	(266,346)	(272,425)	(297,421)	(16,66)	(22,67)	(24,82)	(137,299)
(178,378)	(179,406)	(89,310)	(183,387)	(111,326)	(114,336)	(187,263)	(260,324)	(278,353)	(79,269)
(181,303)	(156,226)	(227,354)	(217,312)	(155,417)	(166,431)	(141,343)	(192,416)	(159,411)	(78,244)
(81,235)	(83,274)	(77,255)	(108,173)	(110,366)	(45,129)	(154,293)	(47,115)	(161,374)	(162,356)
(248,399)	(259,436)	(290,457)	(132,291)	(25,135)	(26,112)	(319,430)	(51,138)	(267,352)	(127,186)
(63,238)	(19,65)	(21,74)	(11,55)	(92,253)	(99,342)	(103,175)	(193,452)	(95,223)	(101,294)
(123,216)	(140,382)	(143,391)	(30,128)	(276,389)	(281,371)	(196,296)	(211,375)	(17,68)	(61,230)

(167,288) (49,184) (72,151) (308,410) (97,305) (218,441) (229,400) (18,70) (20,109) (93,311)
(39,189) (122,277) (125,341) (52,233) (80,283) (280,340) (58,200) (76,286) (243,377) (32,197)
(107,265) (60,164) (71,118) (414,475) (142,408) (84,338) (87,158) (195,404) (198,405) (251,398)
(206,420) (35,117) (185,386) (62,136) (85,258) (145,373) (146,330) (190,370) (202,275) (262,407)
(271,434) (236,435) (318,372) (331,393) (15,64) (133,384) (139,285) (91,174) (247,362) (33,134)
(41,182) (37,222) (199,388) (300,429) (224,327) (168,284) (94,333) (220,469) (245,381) (69,289)
(56,144) (73,150) (306,363) (116,357) (152,402) (214,409) (309,426) (113,169) (264,401) (314,380)
(349,412) (252,392) (149,328) (171,367) (221,466) (228,465) (231,423) (105,213) (106,359) (261,467)
(232,351) (234,460) (119,355) (31,98) (75,322) (207,432) (219,325) (163,396) (172,418) (201,464)
(170,427) (177,438) (287,383) (120,360) (130,364) (304,453) (153,335) (242,477) (246,390) (210,348)
(188,321) (279,455) (298,454) (361,468) (368,462) (301,379) (302,445) (329,428) (307,437) (320,443)
(332,413) (345,450) (240,471) (241,439) (273,433) (250,422) (257,444) (282,470) (337,446) (1,34)
(2,38) (3,121) (4,59) (5,48) (6,176) (7,212) (8,194) (9,46) (10,44) (12,50)
(13,54) (14,126) (23,180) (27,256) (28,148) (36,204) (40,295) (42,225) (43,313) (53,88)
(90,350) (102,369) (131,395) (147,385) (191,339) (203,447) (205,474) (208,476) (239,456) (268,365)
(292,334) (317,449) (344,458) (347,473) (394,442) (403,472) (419,459) (424,463)

- u=242:

(273,433) (290,457) (302,485) (237,448) (249,461) (57,209) (86,316) (124,215) (315,440) (323,415)
(157,376) (165,397) (414,475) (266,346) (272,425) (297,421) (16,66) (22,67) (24,82) (137,299)
(178,378) (179,406) (89,310) (183,387) (111,326) (114,336) (187,263) (260,324) (278,353) (79,269)
(181,303) (156,226) (227,354) (217,312) (155,417) (166,431) (141,343) (192,416) (159,411) (78,244)
(81,235) (10,53) (12,54) (13,131) (110,366) (96,270) (100,368) (47,115) (361,473) (385,479)
(248,399) (212,484) (218,441) (132,291) (25,135) (26,112) (319,430) (51,138) (267,352) (127,186)
(63,238) (19,65) (21,74) (11,55) (92,253) (99,342) (103,175) (193,452) (95,223) (101,294)
(123,216) (104,374) (120,394) (30,128) (276,389) (281,371) (196,296) (211,375) (17,68) (61,230)
(167,288) (49,184) (72,151) (308,410) (97,305) (154,293) (161,419) (18,70) (20,109) (93,311)
(39,189) (122,277) (125,341) (52,233) (80,283) (280,340) (58,200) (76,286) (243,377) (32,197)
(107,265) (60,164) (292,449) (295,482) (142,408) (28,176) (43,121) (195,404) (198,405) (251,398)
(206,420) (35,117) (185,386) (62,136) (85,258) (145,373) (146,330) (190,370) (202,275) (262,407)
(271,434) (236,435) (318,372) (331,393) (15,64) (133,384) (139,285) (91,174) (247,362) (33,134)
(41,182) (37,222) (199,388) (126,379) (148,334) (168,284) (94,333) (220,469) (245,381) (69,289)
(56,144) (73,150) (306,363) (116,357) (191,462) (194,364) (309,426) (113,169) (264,401) (314,380)
(349,412) (252,392) (149,328) (171,367) (221,466) (228,465) (231,423) (105,213) (261,467) (232,351)
(234,460) (119,355) (31,98) (75,322) (207,432) (219,325) (163,396) (172,418) (201,464) (170,427)
(177,438) (287,383) (313,456) (327,459) (304,453) (153,335) (242,477) (246,390) (210,348) (188,321)
(279,455) (298,454) (250,422) (256,424) (257,474) (268,365) (329,428) (307,437) (320,443) (239,483)
(254,451) (240,471) (241,439) (344,458) (347,395) (403,472) (282,470) (337,446) (90,359) (259,436)
(339,468) (332,413) (345,450) (358,478) (1,34) (2,36) (3,38) (4,59) (5,255) (6,71)
(7,44) (8,48) (9,50) (14,140) (23,214) (27,300) (29,224) (40,274) (42,173) (45,152)
(46,84) (77,180) (83,350) (87,158) (88,317) (102,356) (108,147) (118,382) (129,369) (130,301)
(143,391) (160,338) (162,400) (203,463) (204,409) (205,447) (208,402) (225,480) (229,442) (360,444)
(429,476) (445,481)

- u=246:

(358,478) (387,490) (302,485) (237,448) (249,461) (57,209) (86,316) (124,215) (315,440) (323,415)
(157,376) (165,397) (414,475) (266,346) (118,290) (231,423) (274,487) (22,67) (24,82) (137,299)
(178,378) (179,406) (88,143) (102,350) (111,326) (114,336) (187,263) (260,324) (278,353) (79,269)
(181,303) (156,226) (227,354) (217,312) (155,417) (166,431) (141,343) (298,492) (317,488) (78,244)
(81,235) (10,53) (12,54) (84,360) (106,273) (96,270) (100,368) (47,115) (361,473) (385,479)
(248,399) (382,447) (409,493) (132,291) (25,135) (26,112) (319,430) (51,138) (267,352) (127,186)
(63,238) (45,279) (46,87) (11,55) (92,253) (99,342) (103,175) (193,452) (95,223) (101,294)
(123,216) (104,374) (442,489) (30,128) (276,389) (281,371) (196,296) (211,375) (17,68) (61,230)
(167,288) (49,184) (72,151) (308,410) (97,305) (297,421) (345,450) (18,70) (20,109) (93,311)
(39,189) (122,277) (125,341) (52,233) (80,283) (280,340) (58,200) (76,286) (243,377) (32,197)
(107,265) (60,164) (292,449) (295,482) (142,408) (28,176) (43,121) (195,404) (198,405) (251,398)
(206,420) (35,117) (185,386) (62,136) (85,258) (145,373) (146,330) (190,370) (202,275) (262,407)
(271,434) (236,435) (318,372) (331,393) (15,64) (133,384) (139,285) (91,174) (247,362) (33,134)
(41,182) (37,222) (205,455) (208,483) (212,454) (214,443) (224,429) (148,334) (159,411) (245,381)
(69,289) (56,144) (73,150) (306,363) (116,357) (191,462) (194,364) (71,325) (83,239) (160,400)
(161,419) (349,412) (252,392) (149,328) (171,367) (221,466) (229,484) (272,425) (105,213) (261,467)
(232,351) (234,460) (119,355) (31,98) (75,322) (110,366) (126,379) (163,396) (172,418) (201,464)
(170,427) (177,438) (287,383) (313,456) (327,459) (304,453) (153,335) (242,477) (246,390) (210,348)
(188,321) (129,402) (154,293) (300,491) (207,481) (219,486) (220,457) (259,436) (282,470) (307,437)
(339,465) (369,476) (240,471) (329,426) (344,458) (347,395) (403,472) (147,391) (152,356) (23,301)
(27,158) (337,446) (332,413) (218,441) (90,359) (94,333) (310,433) (314,451) (1,34) (2,36)
(3,38) (4,42) (5,254) (6,203) (7,113) (8,268) (9,48) (13,50) (14,131) (16,192)
(19,59) (21,120) (29,250) (40,257) (44,173) (65,183) (66,338) (74,130) (77,255) (89,284)

(140,256) (162,228) (168,445) (169,394) (180,444) (199,388) (204,428) (225,463) (241,439) (264,432)
(309,380) (320,480) (365,401) (416,469) (422,468) (424,474)

• u=250:

(7,113) (8,48) (302,485) (388,444) (424,495) (57,209) (86,316) (124,215) (315,440) (323,415)
(157,376) (165,397) (414,475) (266,346) (118,290) (237,448) (255,494) (264,461) (22,67) (24,82)
(137,299) (178,378) (243,468) (249,487) (102,350) (111,326) (114,336) (187,263) (260,324) (278,353)
(79,269) (181,303) (156,226) (227,354) (217,312) (155,417) (320,498) (141,343) (298,492) (317,488)
(206,433) (220,396) (228,420) (250,464) (84,360) (106,273) (96,270) (100,368) (47,115) (361,473)
(385,479) (248,399) (382,447) (409,493) (132,291) (25,135) (26,112) (319,430) (51,138) (267,352)
(127,186) (63,238) (45,279) (46,87) (11,55) (92,253) (99,342) (103,175) (193,452) (95,223)
(101,294) (123,216) (104,374) (442,489) (30,128) (276,389) (281,371) (196,296) (211,375) (256,422)
(257,490) (309,457) (49,184) (72,151) (308,410) (97,305) (231,496) (274,469) (18,70) (20,109)
(93,311) (39,189) (122,277) (125,341) (52,233) (80,283) (280,340) (58,200) (94,366) (108,225)
(32,197) (107,265) (60,164) (292,449) (295,482) (21,120) (29,310) (43,121) (314,380) (284,497)
(306,363) (333,501) (35,117) (185,386) (62,136) (85,258) (145,373) (146,330) (190,370) (202,275)
(262,407) (271,434) (195,474) (203,427) (244,451) (15,64) (133,384) (139,285) (91,174) (247,362)
(33,134) (41,182) (37,222) (205,455) (208,483) (212,454) (50,171) (77,201) (148,334) (159,411)
(89,345) (131,268) (69,289) (74,179) (76,297) (116,357) (191,462) (194,364) (71,325) (83,239)
(160,400) (161,419) (349,412) (252,392) (149,328) (144,421) (150,387) (229,484) (272,425) (105,213)
(261,467) (232,351) (234,460) (119,355) (31,98) (75,322) (183,432) (192,404) (198,480) (245,381)
(251,398) (259,436) (287,383) (313,456) (327,459) (304,453) (153,335) (140,401) (142,230) (288,391)
(338,445) (129,402) (154,293) (300,491) (207,481) (219,486) (61,177) (68,331) (282,470) (307,437)
(339,465) (166,423) (167,377) (17,56) (19,73) (347,395) (318,478) (358,476) (152,356) (23,301)
(27,158) (337,446) (332,413) (218,441) (90,359) (241,439) (172,416) (348,471) (365,499) (403,472)
(426,477) (1,34) (2,36) (3,38) (4,40) (5,240) (6,286) (9,147) (10,65) (12,54)
(13,242) (14,143) (16,59) (28,78) (42,88) (44,188) (53,130) (66,235) (81,214) (126,390)
(162,367) (163,394) (168,428) (169,435) (170,379) (173,372) (176,429) (180,369) (199,236) (204,450)
(210,406) (221,466) (224,344) (246,463) (254,408) (321,418) (329,443) (393,431) (405,458) (438,500)

• u=254:

(65,162) (97,305) (302,485) (388,444) (424,495) (57,209) (86,316) (124,215) (315,440) (323,415)
(157,376) (165,397) (414,475) (266,346) (118,290) (237,448) (255,494) (264,461) (22,67) (24,82)
(199,408) (204,435) (210,379) (214,431) (102,350) (111,326) (114,336) (187,263) (260,324) (278,353)
(79,269) (155,438) (163,443) (168,393) (235,321) (236,502) (240,394) (141,343) (298,492) (317,488)
(206,433) (220,396) (228,420) (250,464) (84,360) (106,273) (96,270) (100,368) (47,115) (361,473)
(385,479) (248,399) (382,447) (409,493) (132,291) (25,135) (319,430) (51,138) (267,352) (130,329)
(137,390) (143,428) (11,55) (92,253) (99,342) (103,175) (193,452) (95,223) (101,294) (123,216)
(104,374) (442,489) (30,128) (276,389) (281,371) (196,296) (211,375) (256,422) (257,490) (309,457)
(49,184) (72,151) (308,410) (81,176) (88,217) (274,469) (18,70) (20,109) (93,311) (39,189)
(122,277) (125,341) (52,233) (80,283) (280,340) (58,200) (87,146) (110,254) (32,197) (107,265)
(60,164) (202,275) (221,417) (21,120) (29,310) (43,121) (314,380) (284,497) (306,363) (333,501)
(35,117) (185,386) (62,136) (85,258) (145,373) (286,406) (403,472) (426,477) (242,506) (246,508)
(249,487) (203,427) (244,451) (15,64) (133,384) (139,285) (91,174) (247,362) (33,134) (41,182)
(37,222) (205,455) (208,483) (212,454) (50,171) (77,201) (148,334) (159,411) (89,345) (131,268)
(69,289) (74,179) (76,297) (116,357) (191,462) (194,364) (71,325) (83,239) (160,400) (161,419)
(349,412) (252,392) (149,328) (144,421) (150,387) (450,505) (463,509) (105,213) (261,467) (232,351)
(234,460) (119,355) (31,98) (75,322) (183,432) (192,404) (198,480) (245,381) (251,398) (259,436)
(287,383) (313,456) (327,459) (304,453) (153,335) (312,474) (405,482) (288,391) (338,445) (129,402)
(154,293) (300,491) (207,481) (219,486) (61,177) (68,331) (282,470) (307,437) (339,465) (166,423)
(167,377) (17,56) (19,73) (347,395) (318,478) (358,476) (152,356) (23,301) (27,158) (337,446)
(332,413) (218,441) (90,359) (241,439) (172,416) (348,471) (365,499) (299,504) (320,498) (344,458)
(367,500) (369,507) (1,34) (2,36) (3,38) (4,40) (5,42) (6,48) (7,262) (8,78)
(9,126) (10,63) (12,53) (13,188) (14,54) (16,173) (26,226) (28,66) (44,94) (45,229)
(46,330) (59,147) (108,271) (113,156) (127,372) (140,401) (142,295) (169,434) (170,449) (178,407)
(180,466) (181,303) (186,366) (190,425) (195,429) (224,496) (225,370) (227,354) (230,292) (231,418)
(238,484) (243,503) (272,378) (279,468)

• u=258:

(298,492) (300,491) (355,432) (388,444) (424,495) (57,209) (86,316) (124,215) (315,440) (170,425)
(178,468) (181,303) (414,475) (266,346) (118,290) (237,448) (255,494) (264,461) (22,67) (24,82)
(199,408) (204,435) (210,379) (214,431) (102,350) (111,326) (187,263) (260,324) (278,353) (79,269)
(155,438) (163,443) (168,393) (235,321) (236,502) (37,222) (45,229) (46,173) (317,488) (206,433)
(220,396) (228,420) (250,464) (84,360) (106,273) (96,270) (100,368) (47,115) (361,473) (385,479)
(248,399) (382,447) (409,493) (132,291) (25,135) (319,430) (51,138) (267,352) (130,329) (137,390)
(143,428) (11,55) (92,253) (99,342) (103,175) (193,452) (95,223) (101,294) (123,216) (104,374)
(442,489) (30,128) (276,389) (281,371) (196,296) (169,405) (177,426) (257,490) (309,457) (49,184)
(72,151) (308,410) (81,176) (88,217) (274,469) (18,70) (20,109) (93,311) (39,189) (122,277)
(125,341) (52,233) (80,283) (280,340) (58,200) (87,146) (110,254) (32,197) (107,265) (60,164)
(202,275) (144,421) (147,407) (29,310) (43,121) (314,380) (284,497) (306,363) (333,501) (35,117)
(185,386) (62,136) (85,258) (145,373) (192,404) (198,480) (207,481) (227,515) (246,508) (249,487)
(203,427) (244,451) (15,64) (133,384) (139,285) (91,174) (247,362) (33,134) (41,182) (63,180)
(94,186) (108,354) (112,165) (50,171) (77,201) (148,334) (159,411) (89,345) (131,268) (69,289)

(74,179)	(76,297)	(116,357)	(191,462)	(194,364)	(71,325)	(83,239)	(160,400)	(161,419)	(349,412)
(252,392)	(230,517)	(231,466)	(238,401)	(450,505)	(463,509)	(105,213)	(261,467)	(232,351)	(234,460)
(141,343)	(142,403)	(149,328)	(150,387)	(366,472)	(415,477)	(245,381)	(251,398)	(259,436)	(287,383)
(313,456)	(327,459)	(304,453)	(153,335)	(312,474)	(113,397)	(126,279)	(127,272)	(129,402)	(154,293)
(240,394)	(323,512)	(330,513)	(288,391)	(68,331)	(282,470)	(307,437)	(339,465)	(166,423)	(167,377)
(17,56)	(19,73)	(347,395)	(318,478)	(358,476)	(152,356)	(23,301)	(27,158)	(337,446)	(332,413)
(218,441)	(90,359)	(241,439)	(172,416)	(348,471)	(365,499)	(299,504)	(320,498)	(369,507)	(375,482)
(376,496)	(406,503)	(205,455)	(208,483)	(212,454)	(1,34)	(2,36)	(3,38)	(4,40)	(5,42)
(6,44)	(7,48)	(8,195)	(9,78)	(10,53)	(12,54)	(13,221)	(14,243)	(16,66)	(21,61)
(26,271)	(28,295)	(31,188)	(59,225)	(65,344)	(75,322)	(97,211)	(98,262)	(119,338)	(120,190)
(140,372)	(156,445)	(157,256)	(162,434)	(183,417)	(219,484)	(224,510)	(226,422)	(242,506)	(286,486)
(292,514)	(302,418)	(305,485)	(336,511)	(367,500)	(370,458)	(378,429)	(449,516)		

F $(4t, 4, \{3, 5\}, 1)$-CDFs for $t \equiv 1 \pmod 2$, $t \geq 7$ and $t \neq 11$.

Here we list all base blocks of a $(4t, 4, \{3, 5\}, 1)$-CDF for $t \equiv 1 \pmod 2$, $t \geq 7$ and $t \neq 11$.

(1) When $t \equiv 1 \pmod 6$ and $t \geq 7$, the conclusion follows immediately from Lemma 3.5 in [33].

(2) When $t = 29$, a $(4t, 4, \{3, 5\}, 1)$-CDF is listed below:

$$\begin{array}{llllll}
\{0,3,69\}, & \{0,11,86\}, & \{0,16,83\}, & \{0,18,82\}, & \{0,24,77\}, & \{0,1,6,20,27\}, \\
\{0,4,74\}, & \{0,12,85\}, & \{0,17,79\}, & \{0,22,78\}, & \{0,28,76\}, & \{0,2,10,25,61\}, \\
\{0,9,81\}, & \{0,13,84\}. & & & &
\end{array}$$

When $t = 35$, a $(4t, 4, \{3, 5\}, 1)$-CDF is listed below:

$$\begin{array}{llllll}
\{0,1,32,78,104\}, & \{0,4,20\}, & \{0,7,34\}, & \{0,10,96\}, & \{0,13,100\}, & \{0,25,81\}, \\
\{0,3,33,76,98\}, & \{0,5,23\}, & \{0,8,82\}, & \{0,11,102\}, & \{0,14,99\}, & \{0,28,80\}, \\
\{0,2,19\}, & \{0,6,21\}, & \{0,9,92\}, & \{0,12,101\}, & \{0,24,71\}, & \{0,29,79\}.
\end{array}$$

When $t \equiv 5, 11 \pmod{24}$ and $t \geq 53$, take $A_1 = \{0, 1, t-3, 2t+8, 3t-1\}$ and $A_2 = \{0, 3, t-2, 2t+6, 3t-7\}$ as two base blocks with block size five. Let $S = ([1, t-1] \cup [2t+1, 3t-1])$ and $T = \{1, 3, t-13, t-9, t-5, t-4, t-3, t-2, 2t+2, 2t+3, 2t+5, 2t+6, 2t+7, 2t+8, 3t-11, 3t-10, 3t-8, 3t-7, 3t-2, 3t-1\}$. Then we shall show that $S \setminus T$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $1 \leq i \leq (2t-22)/3$, such that $a_i + b_i \equiv c_i \pmod{4t}$.

By Lemma 4.12 with $(d, u, k) = (3, (t-17)/6, (t-17)/3)$, the set $[3, (t-11)/2] \setminus \{(t-13)/2\}$ can be partitioned into triples $\{a'_i, b'_i, c'_i\}$, $1 \leq i \leq (t-17)/6$, such that $a'_i + b'_i = c'_i$. Taking $a_i = 2a'_i$, $b_i = 2b'_i$ and $c_i = 2c'_i$ for $1 \leq i \leq (t-17)/6$, we know that $[6, t-11]_e \setminus \{t-13\}$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $1 \leq i \leq (t-17)/6$, such that $a_i + b_i = c_i$.

Furthermore, $(S \setminus T) \setminus ([6, t-11]_e \setminus \{t-13\})$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $(t-11)/6 \leq i \leq (2t-22)/3$, such that $a_i + b_i \equiv c_i \pmod{4t}$ as follows:

$$\begin{array}{lll}
\multicolumn{3}{l}{\{13+2j, (5t-15)/2-j, (5t+11)/2+j\}, \quad j \in [0, (t-35)/2] \setminus \{(t-37)/2\};} \\
\{4, t-24, t-20\}, & \{t-12, 2t+9, 3t-3\}, & \{2, (5t+1)/2, (5t+5)/2\}, \\
\{11, t-18, t-7\}, & \{t-14, 2t+1, 3t-13\}, & \{(5t-13)/2, (5t-3)/2, t-8\}, \\
\{5, 3t-9, 3t-4\}, & \{9, (5t-9)/2, (5t+9)/2\}, & \{(5t-11)/2, (5t-1)/2, t-6\}, \\
\{t-16, 2t+11, 3t-5\}, & \{7, (5t-7)/2, (5t+7)/2\}, & \{(5t-5)/2, (5t+3)/2, t-1\}, \\
\{t-10, 2t+4, 3t-6\}. & &
\end{array}$$

Then $\{\{0, a_i, c_i\} : 1 \leq i \leq (2t-22)/3\} \cup \{A_1, A_2\}$ forms a $(4t, 4, \{3, 5\}, 1)$-CDF.

(3) When $t = 17$, a $(4t, 4, \{3, 5\}, 1)$-CDF is listed below:

$$\{0,1,5,11,48\}, \quad \{0,3,16,35,44\}, \quad \{0,2,14\}, \quad \{0,7,46\}, \quad \{0,8,50\}, \quad \{0,15,38\}.$$

When $t = 23$, a $(4t, 4, \{3, 5\}, 1)$-CDF is listed below:

$\{0,1,5,11,20\}$, $\{0,7,55\}$, $\{0,12,64\}$, $\{0,17,67\}$, $\{0,21,56\}$, $\{0,22,53\}$,
$\{0,2,49,62,65\}$, $\{0,8,66\}$, $\{0,14,68\}$, $\{0,18,51\}$.

When $t=41$, a $(4t,4,\{3,5\},1)$-CDF is listed below:

$\{0,1,38,90,122\}$, $\{0,2,25\}$, $\{0,7,35\}$, $\{0,11,110\}$, $\{0,15,107\}$, $\{0,18,119\}$,
$\{0,3,39,86,116\}$, $\{0,4,26\}$, $\{0,8,93\}$, $\{0,12,115\}$, $\{0,16,118\}$, $\{0,20,111\}$,
$\{0,19,114\}$, $\{0,5,29\}$, $\{0,9,40\}$, $\{0,13,109\}$, $\{0,17,105\}$, $\{0,34,94\}$,
$\{0,33,100\}$, $\{0,6,27\}$, $\{0,10,108\}$, $\{0,14,120\}$.

When $t\equiv 17,23 \pmod{24}$ and $t\geq 47$, take $A_1=\{0,1,t-3,2t+8,3t-1\}$ and $A_2=\{0,3,t-2,2t+4,3t-7\}$ as two base blocks with block size five. Let $S=([1,t-1]\cup[2t+1,3t-1])$ and $T=\{1,3,t-11,t-9,t-5,t-4,t-3,t-2,2t+1,2t+2,2t+4,2t+5,2t+7,2t+8,3t-11,3t-10,3t-7,3t-6,3t-2,3t-1\}$.

By Lemma 4.11 with $(d,u)=(3,(t-17)/6)$, the set $[3,(t-13)/2]$ can be partitioned into triples $\{a_i',b_i',c_i'\}$, $1\leq i\leq (t-17)/6$, such that $a_i'+b_i'=c_i'$. Taking $a_i=2a_i'$, $b_i=2b_i'$ and $c_i=2c_i'$ for $1\leq i\leq (t-17)/6$, we know that $[6,t-13]_e$ can be partitioned into triples $\{a_i,b_i,c_i\}$, $1\leq i\leq (t-17)/6$, such that $a_i+b_i=c_i$.

Furthermore, $(S\setminus T)\setminus[6,t-13]_e$ can be partitioned into triples $\{a_i,b_i,c_i\}$, $(t-11)/6\leq i\leq (2t-22)/3$, such that $a_i+b_i\equiv c_i \pmod{4t}$ as follows:

$\{11+2j,(5t-13)/2-j,(5t+9)/2+j\}$, $j\in[0,(t-37)/2]$;
$\{7,t-14,t-7\}$, $\{t-12,2t+3,3t-9\}$, $\{5,(5t-5)/2,(5t+5)/2\}$,
$\{4,t-24,t-20\}$, $\{t-22,2t+9,3t-13\}$, $\{(5t-9)/2,(5t-7)/2,t-8\}$,
$\{9,3t-12,3t-3\}$, $\{t-18,2t+10,3t-8\}$, $\{(5t-11)/2,(5t-1)/2,t-6\}$,
$\{t-16,2t+11,3t-5\}$, $\{2,(5t+3)/2,(5t+7)/2\}$, $\{(5t-3)/2,(5t+1)/2,t-1\}$,
$\{t-10,2t+6,3t-4\}$.

Then $\{\{0,a_i,c_i\}:1\leq i\leq (2t-22)/3\}\cup\{A_1,A_2\}$ forms a $(4t,4,\{3,5\},1)$-CDF.

(4) When $t=15$, a $(4t,4,\{3,5\},1)$-CDF is listed below:

$\{0,1,3,12,20\}$, $\{0,4,25\}$, $\{0,5,28\}$, $\{0,6,24\}$, $\{0,7,29\}$, $\{0,10,26\}$, $\{0,13,27\}$.

When $t\equiv 15,21 \pmod{24}$ and $t\geq 21$, take $A=\{0,1,t-2,2t+3,3t-1\}$ as the base block with block size five. Let $S=([1,t-1]\cup[2t+1,3t-1])$ and $T=\{1,t-4,t-3,t-2,2t+1,2t+2,2t+3,3t-5,3t-2,3t-1\}$.

By Lemma 4.12 with $(d,u,k)=(2,(t-9)/6,t/3-3)$, the set $[2,(t-5)/2]\setminus\{(t-7)/2\}$ can be partitioned into triples $\{a_i',b_i',c_i'\}$, $1\leq i\leq (t-9)/6$, such that $a_i'+b_i'=c_i'$. Taking $a_i=2a_i'$, $b_i=2b_i'$ and $c_i=2c_i'$ for $1\leq i\leq (t-9)/6$, we know that $[4,t-5]_e\setminus\{t-7\}$ can be partitioned into triples $\{a_i,b_i,c_i\}$, $1\leq i\leq (t-9)/6$, such that $a_i+b_i=c_i$.

Furthermore, $(S\setminus T)\setminus([4,t-5]_e\setminus\{t-7\})$ can be partitioned into triples $\{a_i,b_i,c_i\}$, $(t-3)/6\leq i\leq 2t/3-4$, such that $a_i+b_i\equiv c_i \pmod{4t}$ as follows:

$\{7+2j,(5t-9)/2-j,(5t+5)/2+j\}$, $j\in[0,(t-17)/2]\setminus\{(t-19)/2\}$;
$\{5,t-12,t-7\}$, $\{t-8,2t+5,3t-3\}$, $\{(5t-7)/2,(5t-5)/2,t-6\}$,
$\{3,3t-7,3t-4\}$, $\{2,(5t-1)/2,(5t+3)/2\}$, $\{(5t-3)/2,(5t+1)/2,t-1\}$.

Then $\{\{0,a_i,c_i\}:1\leq i\leq 2t/3-4\}\cup\{A\}$ forms a $(4t,4,\{3,5\},1)$-CDF.

(5) When $t=9$, a $(4t,4,\{3,5\},1)$-CDF is listed below:

$\{0,1,3,13,17\}$, $\{0,5,11\}$, $\{0,7,15\}$.

When $t\equiv 3,9 \pmod{24}$ and $t\geq 27$, take $A=\{0,1,t-2,2t+3,3t-1\}$ as the base block with block size five. Let $S=([1,t-1]\cup[2t+1,3t-1])$ and $T=\{1,t-4,t-3,t-2,2t+1,2t+2,2t+3,3t-5,3t-2,3t-1\}$.

By Lemma 4.11 with $(d,u)=(2,(t-9)/6)$, the set $[2,(t-7)/2]$ can be partitioned into triples $\{a_i',b_i',c_i'\}$, $1\leq i\leq (t-9)/6$, such that $a_i'+b_i'=c_i'$. Taking $a_i=2a_i'$, $b_i=2b_i'$ and

$c_i = 2c'_i$ for $1 \le i \le (t-9)/6$, we know that $[4, t-7]_e$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $1 \le i \le (t-9)/6$, such that $a_i + b_i = c_i$.

Furthermore, $(S \setminus T) \setminus [4, t-7]_e$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $(t-3)/6 \le i \le 2t/3 - 4$, such that $a_i + b_i \equiv c_i \pmod{4t}$ as follows:

$$\{7+2j, (5t-9)/2-j, (5t+5)/2+j\}, \quad j \in [0, (t-19)/2];$$
$$\{5, t-10, t-5\}, \quad \{t-8, 2t+4, 3t-4\}, \quad \{(5t-7)/2, (5t-5)/2, t-6\},$$
$$\{3, 3t-6, 3t-3\}, \quad \{2, (5t-1)/2, (5t+3)/2\}, \quad \{(5t-3)/2, (5t+1)/2, t-1\}.$$

Then $\{\{0, a_i, c_i\} : 1 \le i \le 2t/3 - 4\} \cup \{A\}$ forms a $(4t, 4, \{3,5\}, 1)$-CDF.

G $(16t, 16, \{3,5\}, 1)$-CDFs for $t \equiv 0 \pmod 2$ and $t \ge 4$

Here we list all base blocks of a $(16t, 16, \{3,5\}, 1)$-CDF for $t \equiv 0 \pmod 2$ and $t \ge 4$.

(1) When $t \equiv 4 \pmod 6$ and $t \ge 4$, the conclusion follows immediately from Lemma 3.8 in [33].

(2) When $t = 12$, a $(16t, 16, \{3,5\}, 1)$-CDF is listed below:

$$\{0,1,34\}, \quad \{0,6,128\}, \quad \{0,11,126\}, \quad \{0,19,142\}, \quad \{0,27,136\}, \quad \{0,30,119\},$$
$$\{0,2,37\}, \quad \{0,7,131\}, \quad \{0,15,140\}, \quad \{0,20,137\}, \quad \{0,28,134\}, \quad \{0,31,44\},$$
$$\{0,3,41\}, \quad \{0,8,105\}, \quad \{0,16,127\}, \quad \{0,21,135\}, \quad \{0,18,130\}, \quad \{0,32,46\},$$
$$\{0,4,43\}, \quad \{0,9,138\}, \quad \{0,17,102\}, \quad \{0,26,139\}, \quad \{0,29,133\}, \quad \{0,25,47,118,141\},$$
$$\{0,5,45\}, \quad \{0,10,110\}, \quad \{0,42,143\}.$$

When $t = 24$, a $(16t, 16, \{3,5\}, 1)$-CDF is listed below:

$$\{0, 11+2j, 244+j\}, \quad j \in [0,16];$$
$$\{0, 53+2j, 266+j\}, \quad j \in [0,17];$$
$$\{0,1,45\}, \quad \{0,5,89\}, \quad \{0,9,91\}, \quad \{0,16,78\}, \quad \{0,26,287\}, \quad \{0,74,265\},$$
$$\{0,2,30\}, \quad \{0,6,40\}, \quad \{0,10,66\}, \quad \{0,18,88\}, \quad \{0,42,284\}, \quad \{0,90,239\},$$
$$\{0,3,54\}, \quad \{0,7,93\}, \quad \{0,12,76\}, \quad \{0,20,263\}, \quad \{0,52,286\}, \quad \{0,92,214\},$$
$$\{0,4,36\}, \quad \{0,8,58\}, \quad \{0,14,94\}, \quad \{0,22,60\}, \quad \{0,68,237\}, \quad \{0,49,95,238,285\}.$$

When $t \equiv 0 \pmod{12}$ and $t \ge 36$, take $A = \{0, 2t+1, 4t-1, 10t-2, 12t-3\}$ as the base block with block size five. Let $S = ([1, 4t-1] \cup [8t+1, 12t-1]) \setminus \{t, 2t, 3t, 9t, 10t, 11t\}$ and $T_1 = \{2t-2, 2t-1, 2t+1, 4t-1, 8t+2, 8t+3, 10t-4, 10t-2, 10t+1, 12t-3\}$.

By Lemma 4.12 with $(d, u, k) = (3, t/6-2, t/12)$, the set $[3, t/2-3] \setminus \{t/4\}$ can be partitioned into triples $\{a'_i, b'_i, c'_i\}$, $1 \le i \le t/6-2$, such that $a'_i + b'_i = c'_i$. Taking $a_i = 4a'_i$, $b_i = 4b'_i$ and $c_i = 4c'_i$ for $1 \le i \le t/6-2$, we have that $T_2 = \{4r : r = 3, 4, \ldots, t/2-3\} \setminus \{t\}$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $1 \le i \le t/6-2$, such that $a_i + b_i = c_i$.

Furthermore, $(S \setminus T_1) \setminus T_2$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $t/6-1 \le i \le 8t/3-6$, such that $a_i + b_i \equiv c_i \pmod{16t}$ as follows:

$$\{11+2j, 10t-7-j, 10t+4+j\}, \quad j \in [0, t-8] \setminus \{2, t/2-5\};$$
$$\{2t+5+2j, 9t-3-j, 11t+2+j\}, \quad j \in [0, t-7];$$
$$\{6+4j, 3t-2-2j, 3t+4+2j\}, \quad j \in [0, t/2-3];$$
$$\{7,8,15\}, \quad \{t+1, 11t-3, 12t-2\}, \quad \{9t-2, 9t-1, 2t-3\},$$
$$\{2,3,5\}, \quad \{2t-8, 2t+3, 4t-5\}, \quad \{10t-9, 10t+2, 4t-7\},$$
$$\{4, 10t-5, 10t-1\}, \quad \{2t+2, 10t-6, 12t-4\}, \quad \{19t/2-2, 21t/2-1, 4t-3\},$$
$$\{9, 10t-3, 10t+6\}, \quad \{2t-4, 10t+3, 12t-1\}, \quad \{8t+1, 11t+1, 3t+2\},$$
$$\{1, 11t-2, 11t-1\}.$$

Then $\{\{0, a_i, c_i\} : 1 \le i \le 8t/3 - 6\} \cup \{A\}$ forms a $(16t, 16, \{3,5\}, 1)$-CDF.

(3) When $t = 14$, a $(16t, 16, \{3,5\}, 1)$-CDF is listed below:

$\{0,1,39\}$, $\{0,6,52\}$, $\{0,11,163\}$, $\{0,17,155\}$, $\{0,26,156\}$, $\{0,33,53\}$,
$\{0,2,43\}$, $\{0,7,123\}$, $\{0,12,47\}$, $\{0,19,147\}$, $\{0,27,151\}$, $\{0,34,165\}$,
$\{0,3,40\}$, $\{0,8,127\}$, $\{0,13,159\}$, $\{0,21,162\}$, $\{0,30,166\}$, $\{0,49,148\}$,
$\{0,4,48\}$, $\{0,9,129\}$, $\{0,15,160\}$, $\{0,36,158\}$, $\{0,32,153\}$, $\{0,29,54,143,161\}$,
$\{0,5,50\}$, $\{0,10,149\}$, $\{0,16,150\}$, $\{0,51,118\}$, $\{0,23,167\}$, $\{0,31,55,142,164\}$.

When $t = 26$, a $(16t, 16, \{3,5\}, 1)$-CDF is listed below:

$\{0, 13+2j, 264+j\}$, $j \in [0,16]$;
$\{0, 57+2j, 287+j\}$, $j \in [0,14]$;
$\{0,1,51\}$, $\{0,7,54\}$, $\{0,14,70\}$, $\{0,20,303\}$, $\{0,74,306\}$, $\{0,96,258\}$,
$\{0,2,38\}$, $\{0,8,94\}$, $\{0,16,60\}$, $\{0,24,309\}$, $\{0,76,231\}$, $\{0,98,281\}$,
$\{0,3,93\}$, $\{0,9,97\}$, $\{0,22,84\}$, $\{0,34,66\}$, $\{0,89,101\}$, $\{0,99,259\}$,
$\{0,5,87\}$, $\{0,10,40\}$, $\{0,4,311\}$, $\{0,68,282\}$, $\{0,92,304\}$, $\{0,53,102,263,305\}$,
$\{0,6,64\}$, $\{0,11,91\}$, $\{0,18,302\}$, $\{0,72,100\}$, $\{0,95,310\}$, $\{0,55,103,262,308\}$.

When $t \equiv 2 \pmod{12}$ and $t \geq 38$, take $A_1 = \{0, 2t+1, 4t-2, 10t+3, 12t-7\}$ and $A_2 = \{0, 2t+3, 4t-1, 10t+2, 12t-4\}$ as two base blocks with block size five. Let $S = ([1,4t-1] \cup [8t+1, 12t-1]) \setminus \{t, 2t, 3t, 9t, 10t, 11t\}$ and $T_1 = \{2t-10, 2t-6, 2t-4, 2t-3, 2t+1, 2t+3, 4t-2, 4t-1, 8t+1, 8t+2, 8t+3, 8t+5, 10t-8, 10t-7, 10t-5, 10t-3, 10t+2, 10t+3, 12t-7, 12t-4\}$.

By Lemma 4.12 with $(d,u,k) = (2, (t-8)/6, (t+10)/12)$, the set $[2, t/2-2] \setminus \{(t+2)/4\}$ can be partitioned into triples $\{a_i', b_i', c_i'\}$, $1 \leq i \leq (t-8)/6$, such that $a_i' + b_i' = c_i'$. Taking $a_i = 4a_i'$, $b_i = 4b_i'$ and $c_i = 4c_i'$ for $1 \leq i \leq (t-8)/6$, we have that $T_2 = \{4r : r = 2, 3, \ldots, t/2-2\} \setminus \{t+2\}$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $1 \leq i \leq (t-8)/6$, such that $a_i + b_i = c_i$.

Furthermore, $(S \setminus T_1) \setminus T_2$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $(t-2)/6 \leq i \leq (8t-28)/3$, such that $a_i + b_i \equiv c_i \pmod{16t}$ as follows:

$\{13+2j, 10t-9-j, 10t+4+j\}$, $j \in [0, t-10] \setminus \{t-11\}$;
$\{2t+5+2j, 9t-4-j, 11t+1+j\}$, $j \in [0, t-10] \setminus \{t/2+4\}$;
$\{6+4j, 3t-2-2j, 3t+4+2j\}$, $j \in [0, t/2-5] \setminus \{(t-6)/4\}$;
$\{3, 12t-8, 12t-5\}$, $\{2t-1, 10t-2, 12t-3\}$, $\{8t+4, 7t/2+1, 23t/2+5\}$,
$\{5, 12t-6, 12t-1\}$, $\{2t+4, 10t-6, 12t-2\}$, $\{10t-4, 10t+1, 4t-3\}$,
$\{2, 4t-11, 4t-9\}$, $\{2t-5, 9t+2, 11t-3\}$, $\{9t-2, 11t-5, 4t-7\}$,
$\{9, 4t-13, 4t-4\}$, $\{t+2, 9t-3, 10t-1\}$, $\{9t-1, 11t-4, 4t-5\}$,
$\{4, 2t+2, 2t+6\}$, $\{1, 11t-2, 11t-1\}$, $\{5t/2+1, 17t/2-8, 11t-7\}$,
$\{7, 2t-9, 2t-2\}$, $\{11, 3t+2, 3t+13\}$.

Then $\{\{0, a_i, c_i\} : 1 \leq i \leq (8t-28)/3\} \cup \{A_1, A_2\}$ forms a $(16t, 16, \{3,5\}, 1)$-CDF.

(4) When $t = 6$, the conclusion follows from Example 2.9. When $t \equiv 6 \pmod{12}$ and $t \geq 18$, take $A = \{0, 2t+1, 4t-1, 10t-2, 12t-3\}$ as the base block with block size five. Let $S = ([1, 4t-1] \cup [8t+1, 12t-1]) \setminus \{t, 2t, 3t, 9t, 10t, 11t\}$ and $T_1 = \{2t-2, 2t-1, 2t+1, 4t-1, 8t+2, 8t+3, 10t-4, 10t-2, 10t+1, 12t-3\}$.

By Lemma 4.12 with $(d,u,k) = (2, t/6-1, (t+6)/12)$, the set $[2, t/2-1] \setminus \{(t+2)/4\}$ can be partitioned into triples $\{a_i', b_i', c_i'\}$, $1 \leq i \leq t/6-1$, such that $a_i' + b_i' = c_i'$. Taking $a_i = 4a_i'$, $b_i = 4b_i'$ and $c_i = 4c_i'$ for $1 \leq i \leq t/6-1$, we have that $T_2 = \{4r : r = 2, 3, \ldots, t/2-1\} \setminus \{t+2\}$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $1 \leq i \leq t/6-1$, such that $a_i + b_i = c_i$.

Furthermore, $(S \setminus T_1) \setminus T_2$ can be partitioned into triples $\{a_i, b_i, c_i\}$, $t/6 \leq i \leq 8t/3-6$, such that $a_i + b_i \equiv c_i \pmod{16t}$ as follows:

$\{9+2j, 10t-6-j, 10t+3+j\}$, $j \in [0, t-7] \setminus \{2\}$;
$\{2t+5+2j, 9t-3-j, 11t+2+j\}$, $j \in [0, t-7] \setminus \{t/2\}$;
$\{10+4j, 3t-4-2j, 3t+6+2j\}$, $j \in [0, t/2-4] \setminus \{(t-10)/4\}$;

$\{6,7,13\}$, $\{2,4t-7,4t-5\}$, $\{2t+2,9t-1,11t+1\}$,
$\{1,3t+4,3t+5\}$, $\{t+2,11t-3,12t-1\}$, $\{8t+1,7t/2+1,23t/2+2\}$,
$\{4,3t-2,3t+2\}$, $\{2t+3,10t-5,12t-2\}$, $\{5t/2+1,17t/2-3,11t-2\}$,
$\{3,10t+2,10t+5\}$, $\{2t-3,10t-1,12t-4\}$, $\{9t-2,11t-1,4t-3\}$,
$\{5,10t-8,10t-3\}$.

Then $\{\{0,a_i,c_i\}:1\le i\le 8t/3-6\}\cup\{A\}$ forms a $(16t,16,\{3,5\},1)$-CDF.

(5) When $t=8$, the conclusion follows from Example 2.10. When $t\equiv 8 \pmod{12}$ and $t\ge 20$, take $A_1=\{0,2t+1,4t-2,10t+3,12t-7\}$ and $A_2=\{0,2t+3,4t-1,10t+2,12t-4\}$ as two base blocks with block size five. Let $S=([1,4t-1]\cup[8t+1,12t-1])\setminus\{t,2t,3t,9t,10t,11t\}$ and $T_1=\{2t-10,2t-6,2t-4,2t-3,2t+1,2t+3,4t-2,4t-1,8t+1,8t+2,8t+3,8t+5,10t-8,10t-7,10t-5,10t-3,10t+2,10t+3,12t-7,12t-4\}$.

By Lemma 4.12 with $(d,u,k)=(2,(t-8)/6,(t+4)/12)$, the set $[2,t/2-2]\setminus\{t/4\}$ can be partitioned into triples $\{a_i',b_i',c_i'\}$, $1\le i\le (t-8)/6$, such that $a_i'+b_i'=c_i'$. Taking $a_i=4a_i'$, $b_i=4b_i'$ and $c_i=4c_i'$ for $1\le i\le (t-8)/6$, we have that $T_2=\{4r:\ r=2,3,\ldots,t/2-2\}\setminus\{t\}$ can be partitioned into triples $\{a_i,b_i,c_i\}$, $1\le i\le (t-8)/6$, such that $a_i+b_i=c_i$.

Furthermore, $(S\setminus T_1)\setminus T_2$ can be partitioned into triples $\{a_i,b_i,c_i\}$, $(t-2)/6\le i\le (8t-28)/3$, such that $a_i+b_i\equiv c_i \pmod{16t}$ as follows:

$\{13+2j,10t-9-j,10t+4+j\}$, $j\in[0,t-10]\setminus\{1,t/2-5\}$;
$\{2t+5+2j,9t-4-j,11t+1+j\}$, $j\in[0,t-10]$;
$\{6+4j,3t-2-2j,3t+4+2j\}$, $j\in[0,t/2-5]$;
$\{4,11,15\}$, $\{2,4t-13,4t-11\}$, $\{10t-2,10t-1,4t-3\}$,
$\{3,2t-5,2t-2\}$, $\{2t+4,10t-6,12t-2\}$, $\{9t-2,11t-5,4t-7\}$,
$\{7,2t-1,2t+6\}$, $\{2t+2,9t-3,11t-1\}$, $\{9t-1,11t-3,4t-4\}$,
$\{1,12t-6,12t-5\}$, $\{t+3,11t-4,12t-1\}$, $\{19t/2-4,21t/2-1,4t-5\}$,
$\{5,12t-8,12t-3\}$, $\{10t-10,10t+1,4t-9\}$, $\{8t+4,11t-2,3t+2\}$,
$\{9,10t-4,10t+5\}$.

Then $\{\{0,a_i,c_i\}:1\le i\le (8t-28)/3\}\cup\{A_1,A_2\}$ forms a $(16t,16,\{3,5\},1)$-CDF.